Characterization of Einstein-Fano manifolds via the Kähler-Ricci flow

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Abstract

We explain a characterization of Einstein-Fano manifolds in terms of the lower bound of the density of the volume of the Kähler-Ricci Flow. This is a direct consequence of Perelman's uniform estimate for the Kähler-Ricci Flow and a C^0 estimate of Tian and Zhu.

1 Introduction

During his visit at the MIT in the spring of 2003 G.Perelman made the following surprising claim. Under the Kähler-Ricci Flow over a Fano manifold the normalized Ricci potential, its gradient and Laplacian, the diameter and the scalar curvature are uniformly bounded. Perelman also gave a sketch of his proof. The proof uses in a crucial way the celebrated Perelman's no local collapsing result. The details have been filled out by Sesum and Tian in [Se-Ti]. By using Perelman's result Tian and Zhu [Ti-Zh] was able to prove the convergence of the Kähler-Ricci flow over solitonic Fano manifolds. In this way they partially prove the important Hamilton-Tian conjecture on the convergence of the Kähler-Ricci flow over Fano manifolds. Perelman's spectacular result combined with the C^0 estimate of Tian and Zhu in [Ti-Zh] implies directly the following characterization of Einstein-Fano manifolds in terms of the lower bound of the density of the volume of the Kähler-Ricci Flow.

Theorem 1 *Let X be a Fano manifold and G be a compact maximal subgroup of the identity component of the group of automorphisms of X. Then X admits a G-invariant Kähler-Einstein metric if and only if the Kähler-Ricci flow $(\omega_t)_t$ with G-invariant initial metric ω satisfies the uniform estimate $\omega_t^n \geq k\,\omega^n$, $k > 0$ for all times $t \geq 0$.*

This is an equivalent form of one of the main resulte in [Ti-Zh]. In writing this fact we took also the occasion to give as much as possible an intrinsic flavor to the proof of the celebrated Yau's C^2 [Yau] and Calabis's C^3-uniform estimates for the complex Monge-Ampère equation in the case of the Kähler-Ricci flow (see also [Cao]).
The first step in proving Perelman's result consist in showing the boundedness of a normalizing constant which appears in the evolution formula of the Ricci potential. Perelman show this by using the monotonicity of his μ functional along the Kähler-Ricci flow. We realize that the boundedness of this constant follows in a classical way by using the generalized Bochner-Kodaira Formula. This leads also to an intresting consequence.

Proposition 1.1 *Along the Kähler-Ricci flow $\frac{d}{dt}\omega_t = \omega_t - \mathrm{Ric}_t = i\partial\bar{\partial}u_t$, $⨍_X e^{-u_t}\omega_t^n = 1$, Perelman's $\mathcal{W}$ functional of the Ricci potential u_t with scale $\tau = 1/2$ is increasing. Moreover the monotonicity is strict unless the flow is a soliton.*

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2 Curvature notions for the tangent bundle

Let (X, J) be a complex manifold of dimension n equipped with a hermitian metric $\omega \in \mathcal{E}(\Lambda^{1,1}_{J} T^*_X)(X)$. We note by $D^{\omega}_{J} = \partial^{\omega} + \bar{\partial}$ the Chern connection of the hermitian tangent bundle $(T_{X,J}, h)$, where $h := \omega(\cdot, J\cdot) - i\omega$ is the hermitian form on $T_{X,J}$ associated to ω. We note by

$$\mathcal{C}_{\omega}(T_{X,J}) := (D^{\omega}_{J})^2 \in \mathcal{E}(\Lambda^{1,1}_{J} T^*_X \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(T_{X,J}))(X)$$

the Chern curvature form, which can also be given by the simpler formula $\mathcal{C}_{\omega}(T_{X,J})\xi = \bar{\partial}\partial^{\omega}\xi$, for any germ of holomorphic vector field $\xi \in \mathcal{O}(T_{X,J})_x$. The Chern curvature $\mathcal{C}^{\omega}_{X,J} \in \mathcal{E}(\mathrm{Herm}(T^{\otimes 2}_{X,J}))(X)$ is the hermitian form on the complex vector bundle $T^{\otimes 2}_{X,J}$ defined by the formula

$$\begin{aligned}\mathcal{C}^{\omega}_{X,J}(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) &:= h(\mathcal{C}_{\omega}(T_{X,J})(\xi^{1,0}_1, \xi^{0,1}_2)\eta_1, \eta_2)\\ &= -2i\omega(\mathcal{C}_{\omega}(T_{X,J})(\xi^{1,0}_1, \xi^{0,1}_2)\eta^{1,0}_1, \eta^{0,1}_2)\end{aligned}$$

for all real vector fields $\xi_j, \eta_j \in \mathcal{E}(T_X)(U)$, $j = 1, 2$ on some open subset U. The Griffiths curvature is defined by the formula

$$G^{\omega}_{X,J}(\xi \otimes \eta) := \mathcal{C}^{\omega}_{X,J}(\xi \otimes \eta, \xi \otimes \eta) = \frac{1}{2} h(J\mathcal{C}_{\omega}(T_{X,J})(\xi, J\xi)\eta, \eta).$$

The fact that the Chern curvature is a hermitian form implies that the Griffiths curvature takes allways real values. Then we deduce the identity

$$2G^{\omega}_{X,J}(\xi \otimes \eta) = \omega(\mathcal{C}_{\omega}(T_{X,J})(\xi, J\xi)\eta, \eta). \tag{1}$$

If $\xi, \eta \in \mathcal{O}(T_{X,J})(U)$ are holomorphic vector fields then the Griffiths curvature can be given by the simple formula

$$G^{\omega}_{X,J}(\xi \otimes \eta) = -\xi^{1,0}.\,\xi^{0,1}.\,|\eta|^2_{\omega} + |\partial^{\omega}_{\xi}\,\eta|^2_{\omega}\,.$$

(see for example [Kob]). Let $(z_1, ..., z_n)$ be holomorphic coordinates and let $(\zeta_k)_k \in \mathcal{O}(T^{1,0}_{X,J})^{\oplus n}(U)$ be a local holomorphic frame of the vector bundle $T^{1,0}_{X,J}$. Consider the local expression of the metric $\omega = \frac{i}{2}\sum_{k,l} \omega_{k,\bar{l}}\, \zeta^*_k \wedge \bar{\zeta}^*_l$, where the coefficients $\omega_{k,\bar{l}}$ satisfient the hermitian symmetry relation $\overline{\omega_{k,\bar{l}}} = \omega_{l,\bar{k}}$. We note by $(\omega^{k,\bar{l}}) = (\omega_{k,\bar{l}})^{-1}$ the inverse matrix of $(\omega_{k,\bar{l}})$, namely $\sum_t \omega^{k,\bar{t}}\omega_{t,\bar{l}} = \delta_{k,l}$. If $\alpha \in \Lambda^{p,q}_{J} T^*_X \otimes_{\mathbb{C}} T^{1,0}_{X,J}$ then we will note by $\alpha \otimes_{J} \zeta^*_m := \alpha \otimes \zeta^*_m + \overline{\alpha(\cdot)} \otimes \bar{\zeta}^*_m$. With this notations the Chern curvature form is given locally by the expresion

$$\begin{aligned}\mathcal{C}_{\omega}(T_{X,J}) &= \sum_{m=1}^{n} (\bar{\partial}\partial^{\omega}\zeta_m) \otimes_{J} \zeta^*_m = \sum_{l,m=1}^{n} C_{l,m} \otimes \zeta^*_m \otimes_{J} \zeta_l\\ &= \sum_{j,k,l,m=1}^{n} C^{j,\bar{k}}_{l,m}\,(dz_j \wedge d\bar{z}_k) \otimes \zeta^*_m \otimes_{J} \zeta_l,\end{aligned}$$

with

$$C_{l,m} := -\sum_{r=1}^{n}\Big(\partial\bar\partial\omega_{m,\bar r} - \sum_{s,t=1}^{n}\partial\omega_{m,\bar s}\wedge\omega^{s,\bar t}\bar\partial\omega_{t,\bar r}\Big)\omega^{r,\bar l}. \tag{2}$$

The Chern curvature have the local expression

$$\mathcal{C}^{\omega}_{X,J} = \sum_{j,k,l,m=1}^{n} C_{j,l,\bar k,\bar m}\, dz_j\otimes\zeta_l^*\otimes d\bar z_k\otimes\bar\zeta_m^*,$$

where the coefficients $C_{j,l,\bar k,\bar m} := \sum_{h=1}^{n} C_{h,l}^{j,\bar k}\cdot\omega_{h,\bar m}$ satisfient the hermitian symmetry relation $\overline{C_{j,l,\bar k,\bar m}} = C_{k,m,\bar j,\bar l}$. The following lemma shows that the Chern curvature is the obstruction to the existence of holomorphic frames orthonormed at an order higher than one.

Lemma 1 *Let (X,J) be a complex manifold of dimension n equipped with a hermitian metric $\omega\in\mathcal{E}(\Lambda^{1,1}_{J}T^*_X)(X)$. Then for every point $x\in X$ and any $\omega(x)$-orthonormed frame $(e_k)_k\subset T^{1,0}_{X,J,x}$ there exists holomorphic coordinates $(z_1,...,z_n)$ centered at x and an holomorphic frame $(\zeta_k)_k\in\mathcal{O}(T^{1,0}_{X,J})^{\oplus n}(U_x)$, $\zeta_k(x)=e_k$, in a neighborhood of x such that the metric ω have the local expression*

$$\omega = \frac{i}{2}\sum_l \zeta_l^*\wedge\bar\zeta_l^* - \frac{i}{2}\sum_{j,k,l,m} H_{l,\bar m}^{j,\bar k}\, z_j\bar z_k\,\zeta_l^*\wedge\bar\zeta_m^* + O(|z|^3),$$

where the coefficients $H_{l,\bar m}^{j,\bar k}$ satisfient the hermitian symmetry $\overline{H_{l,\bar m}^{j,\bar k}} = H_{m,\bar l}^{k,\bar j}$. Moreover for any such coordinates and frames the Chern curvatures have at the point x the expressions

$$\mathcal{C}_\omega(T_{X,J})(x) = \sum_{j,k,l,m=1}^{n} H_{m,\bar l}^{j,\bar k}\,(dz_j\wedge d\bar z_k)\otimes\zeta_m^*\otimes_J\zeta_l,$$

$$\mathcal{C}^{\omega}_{X,J}(x) = \sum_{j,k,l,m=1}^{n} H_{l,\bar m}^{j,\bar k}\, dz_j\otimes\zeta_l^*\otimes d\bar z_k\otimes\bar\zeta_m^*.$$

We define the Ricci tensor $\mathrm{Ric}_J(\omega)\in\mathcal{E}(\Lambda^{1,1}_J T^*_X\cap\Lambda^2_{\mathbb{R}}T^*_X)(X)$ of the metric ω respect to the complex structure J by the formula

$$\mathrm{Ric}_J(\omega) := i\,\mathrm{Tr}_{\mathbb{C}}\,\mathcal{C}_\omega(T_{X,J}) = i\,\mathcal{C}_\omega(K^{-1}_{X,J})\in 2\pi c_1(X)\,,$$

where $\mathcal{C}_\omega(K^{-1}_{X,J})$ is the Chern curvature form of the anticanonical bundle $K^{-1}_{X,J} := \Lambda^n_{\mathbb{C}}T_{X,J}$. The scalar curvature $\mathrm{Sc}_J(\omega)\in\mathcal{E}(X,\mathbb{R})$ of ω respect to J is defined by the formula

$$\mathrm{Sc}_J(\omega) := \mathrm{Tr}_\omega(\mathrm{Ric}_J(\omega)) = \frac{2n\,\mathrm{Ric}_J(\omega)\wedge\omega^{n-1}}{\omega^n}.$$

The fact that the Chern connection is invariant by scalar multiplications of the metric implies that $\mathrm{Ric}_J(\lambda\omega) = \mathrm{Ric}_J(\omega)$ for every real number $\lambda>0$. The Ricci curvature have the following local expression

$$\mathrm{Ric}_J(\omega) = i\sum_{1\le j,k,l\le n} C_{l,l}^{j,\bar k}\,\zeta_j^*\wedge\bar\zeta_k^*.$$

We remind (cf. [Dem]) that if $(L, h) \to (X, J)$ is a holomorphic hermitian line bundle and $\sigma \in \mathcal{O}(L \smallsetminus 0)(U)$ is a non vanishing holomorphic section over an open set U then the local expression of the Chern curvature is given by the formula

$$\mathcal{C}_h(L) = -\partial\bar{\partial} \log |\sigma|_h^2$$

on U. If $(\zeta_k)_k \in \mathcal{O}(T_{X,J}^{1,0})^{\oplus n}(U)$ is a local holomorphic frame of the vector bundle $T_{X,J}^{1,0}$ then $|\zeta_1 \wedge ... \wedge \zeta_n|_\omega^2 = \det(\omega_{k,\bar{l}})$. We deduce that the local expression of the Ricci curvature is given by the formula

$$\operatorname{Ric}_J(\omega) = -i\partial\bar{\partial} \log \det(\omega_{k,\bar{l}}).$$

If ω_1 is an other J-invariant metric then we have the global identity

$$\operatorname{Ric}_J(\omega_1) - \operatorname{Ric}_J(\omega) = -i\partial\bar{\partial} \log\left(\frac{\omega_1^n}{\omega^n}\right).$$

2.1 The Kähler case

If (X, J, ω) is a Kähler manifold then the Chern connection coincides with the Levi-Civita connection of the J-invariant Riemannian metric $g \equiv g_{\omega,J}$ associated to ω. This implies that in the Kähler case the Chern curvature form coincides with the Riemann curvature form $\mathcal{R}_g$. In this case the Riemann curvature

$$R_g(\xi, \eta, \mu, \zeta) \equiv R_g(\xi \wedge \eta, \mu \wedge \zeta) := g(\mathcal{C}_\omega(T_{X,J})(\xi, \eta)\zeta, \mu),$$

$(\xi, \eta, \mu, \zeta \in T_X)$ is a smooth section of the vector bundle $S_{\mathbb{R}}^2(\Lambda_J^{1,1} T_X^* \cap \Lambda_{\mathbb{R}}^2 T_X^*)$. We consider also the $\mathbb{C}$-linear extension of the Riemann curvature on the complexified tangent bundle $T_X \otimes_{\mathbb{R}} \mathbb{C}$. We have the equalities

$$R_g(\xi_1^{1,0}, \xi_2^{0,1}, \eta_1^{1,0}, \eta_2^{0,1}) = -R_g(\xi_1^{1,0}, \xi_2^{0,1}, \eta_2^{0,1}, \eta_1^{1,0}) =$$

$$= i\omega(\mathcal{C}_\omega(T_{X,J})(\xi_1^{1,0}, \xi_2^{0,1})\eta_1^{1,0}, \eta_2^{0,1}) = -\frac{1}{2}\mathcal{C}_{X,J}^\omega(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2).$$

The Riemann curvature have the following local expression in arbitrary holomorphic coordinates $(z_1, ..., z_n)$

$$R_g = \sum_{j,k,l,m=1}^n R_{j,\bar{k},l,\bar{m}}\, (dz_j \wedge d\bar{z}_k) \otimes (dz_l \wedge d\bar{z}_m),$$

where the coefficients are given by the formula $2R_{j,\bar{k},l,\bar{m}} = -C_{j,l,\bar{k},\bar{m}}$ respect to the frame $(\zeta_j) := (\partial/\partial z_j)$. Using the formula (2) respect to the frame (ζ_j) we deduce the expression

$$2R_{j,\bar{k},l,\bar{m}} = \frac{\partial^2 \omega_{l,\bar{m}}}{\partial z_j \partial \bar{z}_k} - \sum_{s,t=1}^n \frac{\partial \omega_{l,\bar{s}}}{\partial z_j}\, \omega^{s,\bar{t}}\, \frac{\partial \omega_{t,\bar{m}}}{\partial \bar{z}_k}. \tag{3}$$

The facts that the Riemann curvature is real, is symmetric over $\Lambda_J^{1,1} T_X^*$ and the first Bianchi identity $\mathcal{R}_g(\xi, \eta)\mu + \mathcal{R}_g(\eta, \mu)\xi + \mathcal{R}_g(\mu, \xi)\eta = 0$, are expressed in terms of the coefficients of the Riemann curvature by the symmetries

$$\overline{R_{j,\bar{k},l,\bar{m}}} = R_{k,\bar{j},m,\bar{l}}$$
$$R_{j,\bar{k},l,\bar{m}} = R_{l,\bar{m},j,\bar{k}}$$
$$R_{j,\bar{k},l,\bar{m}} = R_{j,\bar{m},l,\bar{k}},$$

(the second and last equality implies also $R_{j,\bar{k},l,\bar{m}} = R_{l,\bar{k},j,\bar{m}}$). By the other hand we see that the second and last equality follows immediately from the Kähler symmetries

$$\frac{\partial \omega_{l,\bar{m}}}{\partial z_j} = \frac{\partial \omega_{j,\bar{m}}}{\partial z_l}, \qquad \frac{\partial \omega_{l,\bar{m}}}{\partial \bar{z}_k} = \frac{\partial \omega_{l,\bar{k}}}{\partial \bar{z}_m}.$$

Holomorphic geodesic coordinates. In the Kähler case the conclusions of lemma 1 holds for the frame $(\zeta_j) := (\partial/\partial z_j)$. In fact we have the following strongest result.

Lemma 2 *Let (X, J, ω) be a Kähler manifold of dimension n. Then for every point $x \in X$ and any $\omega(x)$-orthonormed frame $(e_k)_k \subset T^{1,0}_{X,J,x}$ there exist holomorphic coordinates $(z_1, ..., z_n)$ centered at x such that $\frac{\partial}{\partial z_k}|_x = e_k$ and the metric ω have the local expression*

$$\omega = \frac{i}{2}\sum_l dz_l \wedge d\bar{z}_l - \frac{i}{2}\sum_{j,k,l,m} H^{j,\bar{k}}_{l,\bar{m}}\, z_j \bar{z}_k\, dz_l \wedge d\bar{z}_m + O(|z|^3),$$

where the coefficients $H^{j,\bar{k}}_{l,\bar{m}}$ satisfient the symmetries $\overline{H^{j,\bar{k}}_{l,\bar{m}}} = H^{k,\bar{j}}_{m,\bar{l}}$, and $H^{j,\bar{k}}_{l,\bar{m}} = H^{l,\bar{k}}_{j,\bar{m}} = H^{j,\bar{m}}_{l,\bar{k}}$. Moreover for any such coordinates the Chern curvatures have at the point x the expressions

$$\mathcal{C}_\omega(T_{X,J})(x) = \sum_{j,k,l,m} H^{j,\bar{k}}_{m,\bar{l}}\,(dz_j \wedge d\bar{z}_k) \otimes dz_m \otimes_J \frac{\partial}{\partial z_l},$$
$$\mathcal{C}^\omega_{X,J}(x) = \sum_{j,k,l,m} H^{j,\bar{k}}_{l,\bar{m}}\, dz_j \otimes dz_l \otimes d\bar{z}_k \otimes d\bar{z}_m.$$

The lemma shows that in the Kähler case the Chern curvature is the obstruction to the existence of holomorphic coordinates $(z_1, ..., z_n)$ such that the frame $(\partial/\partial z_j)$ is orthonormed at an order higher than one. This coordinates are called geodesic holomorphic coordinates.
It will also be usefull a more precise version of the lemma 2. We need first some notation. Consider the complex vector bundle $F := S^2_{\mathbb{C}}\Lambda^{1,1}_J T^*_X$ equipped with the connection ∇_F induced by the complexified Levi-Civita connection. We have the following lemma.

Lemma 3 *Let (X, J, ω) be a Kähler manifold of dimension n. Then for every point $x \in X$ and any $\omega(x)$-orthonormed frame $(e_k)_k \subset T^{1,0}_{X,J,x}$ there exist ω-geodesic holomorphic coordinates $(z_1, ..., z_n)$ centered at x such that $\frac{\partial}{\partial z_k}|_x = e_k$ and the metric ω have the local expression $\omega = \frac{i}{2}\sum_{l,m}\omega_{l,\bar{m}}\, dz_l \wedge d\bar{z}_m$, with*

$$\omega_{l,\bar{m}} = \delta_{l,\bar{m}} - \sum_{j,k} H^{j,\bar{k}}_{l,\bar{m}}\, z_j \bar{z}_k - \sum_{p,j,k}\left(H^{p,j,\bar{k}}_{l,\bar{m}}\, z_p z_j \bar{z}_k + \overline{H^{p,j,\bar{k}}_{m,\bar{l}}}\, z_k \bar{z}_p \bar{z}_j\right) + O(|z|^4),$$

where the coefficients $H^{j,\bar{k}}_{l,\bar{m}}$ satisfient the symmetries of lemma 2 and the coefficients $H^{p,j,\bar{k}}_{l,\bar{m}}$ are symmetric in the indexes p, j, l and k, m. Moreover for any

such coordinates the Riemann curvature and its first covariant derivatives has at the point x the expressions

$$R_\omega(x) = \frac{1}{2}\sum_{j,k,l,m} H^{j,\bar{k}}_{l,\bar{m}}\,(dz_j \wedge d\bar{z}_k) \otimes (dz_l \wedge d\bar{z}_m),$$

$$\nabla^{1,0}_F R_\omega(x) = \frac{1}{2}\sum_{j,k,l,m,p} H^{p,j,\bar{k}}_{l,\bar{m}}\, dz_p \otimes (dz_j \wedge d\bar{z}_k) \otimes (dz_l \wedge d\bar{z}_m),$$

$$\nabla^{0,1}_F R_\omega(x) = \frac{1}{2}\sum_{j,k,l,m,p} \overline{H^{p,k,\bar{j}}_{m,\bar{l}}}\, d\bar{z}_p \otimes (dz_j \wedge d\bar{z}_k) \otimes (dz_l \wedge d\bar{z}_m).$$

The Bisectional curvature. Consider now the bisectional curvature

$$b\sigma_g(\xi,\eta) := R_g(\xi, J\xi, \eta, J\eta) = 4R_g(\xi^{1,0}, \xi^{0,1}, \eta^{0,1}, \eta^{1,0}),$$

(the last equality follows from the identity $\xi \wedge J\xi = -2i\xi^{1,0} \wedge \xi^{0,1}$). We remark that the bisectional curvature coincides with the sectional curvature $\sigma_g(\xi,\eta) := R_g(\xi,\eta,\xi,\eta)$ on complex lines, (in fact $\sigma_g(\xi, J\xi) = b\sigma_g(\xi,\xi)$). The identity (1) shows that in the Kähler case the Griffiths curvature coincides (modulo a factor 2) with the bisectional curvature. In the Kähler case the Riemann curvature is determined by the bisectional curvature. In fact the vector bundle $\Lambda^{1,1}_J T_X$ is generated over $\mathbb{C}$ by the vectors of type $\xi^{1,0} \wedge \xi^{0,1}$, (see for example [Dem], Chapter III, sect 1).

The Riemann curvature operator. Let $G \in \mathcal{E}(S^2_{\mathbb{R}}(\Lambda^{1,1}_J T^*_X \cap \Lambda^2_{\mathbb{R}} T^*_X))(X)$ be the induced metric over the real vector bundle $\Lambda^{1,1}_J T_X \cap \Lambda^2_{\mathbb{R}} T_X$. We will still note by $G \in \mathcal{E}(S^2_{\mathbb{C}}(\Lambda^{1,1}_J T^*_X))(X)$ the $\mathbb{C}$-linear extension over the complexified vector bundle $\Lambda^{1,1}_J T_X$. Explicitly the metric G is given by the formula

$$G(u_1 \wedge u_2, v_1 \wedge v_2) := \det(g(u_k, v_l))_{k,l} = \omega(u_1, v_2) \cdot \omega(v_1, u_2),$$

for any $u_1, v_1 \in T^{1,0}_{X,J}$ and $u_2, v_2 \in T^{0,1}_{X,J}$. We remind now that the Riemann curvature operator $\mathrm{Rm}_g \in \mathcal{E}(\mathrm{End}_{\mathbb{R}}(\Lambda^{1,1}_J T_X \cap \Lambda^2_{\mathbb{R}} T_X))(X)$ is defined by the formula

$$G(\mathrm{Rm}_g(\xi \wedge \eta), \mu \wedge \zeta) := R_g(\xi \wedge \eta, \mu \wedge \zeta),$$

for any $\xi, \mu \in T^{1,0}_{X,J}$ and $\eta, \zeta \in T^{0,1}_{X,J}$. In local coordinates we find the expression

$$\mathrm{Rm}_g = \sum_{j,k,s,t=1}^{n} \mathrm{Rm}^{s,\bar{t}}_{j,\bar{k}} \Big(dz_j \wedge d\bar{z}_k\Big) \otimes \Big(\frac{\partial}{\partial z_s} \wedge \frac{\partial}{\partial \bar{z}_t}\Big),$$

with

$$\mathrm{Rm}^{s,\bar{t}}_{j,\bar{k}} = -4 \sum_{l,m=1}^{n} \omega^{t,\bar{l}} \omega^{m,\bar{s}} R_{j,\bar{k},l,\bar{m}}.$$

So in conclusion if put $H := (\omega_{k,\bar{l}})$, we have the following synthetic expression

$$\mathrm{Rm}_g = 2\sum_{s,t=1}^{n} H^{-1}\bar{\partial}(\partial H \cdot H^{-1})_{t,s} \otimes \Big(\frac{\partial}{\partial z_s} \wedge \frac{\partial}{\partial \bar{z}_t}\Big)$$

for the curvature operator. The fact that Rm_g is a real operator implies the conditions

$$\overline{\mathrm{Rm}^{s,\bar{t}}_{j,\bar{k}}} = \mathrm{Rm}^{t,\bar{s}}_{k,\bar{j}}\,.$$

The Ricci tensor. We remind that in the Riemannian case the Ricci curvature $\mathrm{Ric}(g) \in \mathcal{E}(S^2_{\mathbb{R}} T^*_X)(X)$ is defined by the formula

$$\mathrm{Ric}(g)(\xi, \eta) := \mathrm{Tr}_{\mathbb{R}}(\mathcal{R}_g(\cdot, \xi)\eta)$$

for every $\xi, \eta \in T_X$. If (X, J, ω) is a Kähler manifold and g is the J-invariant Riemannian metric associated to ω, then we have the formula

$$\mathrm{Ric}_{_J}(\omega)(\xi, J\eta) = \mathrm{Ric}(g)(\xi, \eta)$$

for every $\xi, \eta \in T_X$. Let $(z_1, ..., z_n)$ be ω-geodesic coordinates centered in a point x and set $\omega_0 := \frac{i}{2} \sum_k dz_k \wedge d\bar{z}_k$ and write $\mathrm{Ric}_{_J}(\omega) = i \sum_{k,l} R_{k\bar{l}}\, dz_k \wedge d\bar{z}_l$. Then we have the expansion

$$\omega^n = \left(1 - \sum_{k,l} R_{k\bar{l}}(x) z_k \bar{z}_l \right) \omega_0^n + O(|z|^3)\,. \tag{4}$$

Starting from next section we will allways use Einstein's convention of sums.

3 The generalized Bochner-Kodaira formula for compact Kähler manifolds

In writing this section we was inspired by [Fu]. Let (X, ω) be a compact Kähler manifold of complex dimension n and let

$$\langle \alpha, \beta \rangle_\omega := \mathrm{Tr}_\omega(i\alpha \wedge \bar{\beta})/2 = \frac{n\, i\alpha \wedge \bar{\beta} \wedge \omega^{n-1}}{\omega^n},$$

be the induced hermitian product over the complex vector bundle $\Lambda^{1,0}_J T^*_X$. Moreover let $h \in \mathcal{E}(X, \mathbb{R})$ and $u \in \mathcal{E}(X, \mathbb{C})$ be smooth functions. Then the Laplacian $\Delta_{\omega,h} u := \Delta_\omega u + 2 \langle \partial u, \partial h \rangle_\omega$ is a self-adjoint differential operator respect to the inner product defined by the weighted volume form $e^h \omega^n$:

$$(u, v)_{\omega,h} := \int\limits_X u\bar{v}\, e^h \omega^n.$$

In fact this follows from the identities $(\Delta_{\omega,h} u) e^h = -\mathrm{Tr}_\omega \left[i\bar{\partial}(e^h \partial u) \right]$ and

$$-\int\limits_X i\bar{\partial}(e^h \partial u)\bar{v} \wedge \omega^{n-1} = -\int\limits_X i\partial u \wedge \bar{\partial}\bar{v} \wedge e^h \omega^{n-1} = \int\limits_X u\, i\partial(e^h \bar{\partial}\bar{v}) \wedge \omega^{n-1}.$$

We say that $\lambda \in \mathbb{C}$ is an eigenvalue of $\Delta_{\omega,h}$ if there exists a function $u \in \mathcal{E}(X, \mathbb{C})$, not identically zero, such that $\Delta_{\omega,h} u + \lambda u = 0$. Since

$$-\int\limits_X (\Delta_{\omega,h} u)\, \bar{u}\, e^h \omega^n = \int\limits_X 2|\partial u|^2_\omega\, e^h \omega^n,$$

for any $u \in \mathcal{E}(X, \mathbb{C})$, all the eigenvalues of $\Delta_{\omega,h}$ are nonnegative real numbers.

Lemma 4 (Generalized Bochner-Kodaira formula). *Let (X,ω) be a compact Kähler manifold of complex dimension n and let $u,\, h\in\mathcal{E}(X,\mathbb{R})$ be smooth real functions. Then we have the Bochner type formula*

$$\begin{aligned}\int\limits_X |\bar\partial\,\nabla^{1,0}_\omega u|^2_\omega\, e^h\omega^n &= -\int\limits_X \left\langle \partial\Delta_{\omega,h}u,\partial u\right\rangle_\omega e^h\omega^n\\ &- \int\limits_X \big(\mathrm{Ric}(\omega)-i\partial\bar\partial h\big)\left(\nabla_\omega u, J\nabla_\omega u\right) e^h\omega^n. \qquad (5)\end{aligned}$$

Proof. Let $(z_1,...,z_n)$ be ω-geodesic holomorphic coordinates with center a point x. By definition of the $(2,0)$-component of the Hessian we have the identity $\nabla^{1,0}_\omega\partial u\,(\xi,\eta)=\xi.\,\eta.\,u-(\nabla^{1,0}_{\omega,\xi}\,\eta).\,u$ for every $(1,0)$-vector field $\xi,\eta\in\mathcal{E}(T^{1,0}_{X,J})(U)$ over an open set U. By using the equality $\nabla^{1,0}_\omega\frac{\partial}{\partial z_l}=\partial\omega_{l,\bar j}\,\omega^{j,\bar k}\otimes\frac{\partial}{\partial z_k}$, we deduce the local expression

$$\begin{aligned}\nabla^{1,0}_\omega\partial u &= \Big(u_{k,l}-\frac{\partial\omega_{l,\bar j}}{\partial z_k}\,\omega^{j,\bar r}\,u_r\Big)\,dz_k\otimes dz_l\\ &= \Big(u_{k,l}+C^{k,\bar t}_{r,l}\,\bar z_t\,u_r\Big)\;dz_k\otimes dz_l+O(|z|^2).\end{aligned}$$

Moreover the local expression $\nabla^{1,0}_\omega u=2u_{\bar k}\frac{\partial}{\partial z_k}+O(|z|^2)$ implies the local expression

$$\begin{aligned}\nabla^{1,0}_\omega u\,\lrcorner\,\nabla^{1,0}_\omega\partial u &= 2(u_{k,l}u_{\bar k}+C^{k,\bar t}_{r,l}\,\bar z_t\,u_r u_{\bar k})\,dz_l+O(|z|^2)\\ &= 2(u_{k,l}u_{\bar k}+C^{k,\bar r}_{t,l}\,\bar z_t\,u_{\bar k}u_r)\,dz_l+O(|z|^2).\end{aligned}$$

We deduce the equality at the point x

$$\begin{aligned}-\operatorname{Tr}_\omega\Big[i\bar\partial(\nabla^{1,0}_\omega u\,\lrcorner\,\nabla^{1,0}_\omega\partial u)\Big](x) &= 8(u_{k,l}u_{\bar k})_{\bar l}+8C^{k,\bar r}_{l,l}\,u_{\bar k}u_r\\ &= 8(u_{k,l}u_{\bar k})_{\bar l}-2i\,\mathrm{Ric}(\omega)(\nabla^{1,0}_\omega u,\nabla^{0,1}_\omega u)\\ &= 8(u_{k,l}u_{\bar k})_{\bar l}+\mathrm{Ric}(\omega)(\nabla_\omega u,J\nabla_\omega u)(x). \qquad (6)\end{aligned}$$

Consider now the trivial equalities at the point x

$$\begin{aligned}|\bar\partial\,\nabla^{1,0}_\omega u|^2_\omega(x)=8\,u_{k,l}u_{\bar k,\bar l} &= 8\,(u_{k,l}u_{\bar k})_{\bar l}-8\,u_{k,l,\bar l}\,u_{\bar k}\\ &= 8(u_{k,l}u_{\bar k})_{\bar l}-\left\langle\partial\Delta_\omega u,\partial u\right\rangle_\omega(x).\end{aligned}$$

Then using the equality (6) and the identity

$$\left\langle\partial\Delta_{\omega,h}u,\partial u\right\rangle_\omega=\left\langle\partial\Delta_\omega u,\partial u\right\rangle_\omega+2\nabla^{1,0}_\omega\partial u(\nabla^{1,0}_\omega u,\nabla^{1,0}_\omega h)+i\partial\bar\partial h(\nabla_\omega u,J\nabla_\omega u),$$

we deduce the formula

$$\begin{aligned}|\bar\partial\,\nabla^{1,0}_\omega u|^2_\omega &= -\left\langle\partial\Delta_{\omega,h}u,\partial u\right\rangle_\omega-\big(\mathrm{Ric}(\omega)-i\partial\bar\partial h\big)\left(\nabla_\omega u,J\nabla_\omega u\right)\\ &- \operatorname{Tr}_\omega\Big[i\bar\partial\left(\nabla^{1,0}_\omega u\,\lrcorner\,\nabla^{1,0}_\omega\partial u\right)\Big]+2\nabla^{1,0}_\omega\partial u\left(\nabla^{1,0}_\omega u,\nabla^{1,0}_\omega h\right). \qquad (7)\end{aligned}$$

Moreover consider the equality

$$\begin{aligned}2n\,\bar\partial\Big[\left(i\nabla^{1,0}_\omega u\,\lrcorner\,\nabla^{1,0}_\omega\partial u\right)\wedge e^h\omega^{n-1}\Big] &= 2n\,i\bar\partial\left(\nabla^{1,0}_\omega u\,\lrcorner\,\nabla^{1,0}_\omega\partial u\right)\wedge e^h\omega^{n-1}\\ &- 2n\left(\nabla^{1,0}_\omega u\,\lrcorner\,\nabla^{1,0}_\omega\partial u\right)\wedge i\bar\partial h\wedge e^h\omega^{n-1}.\end{aligned}$$

The last term is equal to

$$\begin{aligned}
&- && 2n\left(\nabla^{1,0}_\omega u \lrcorner \nabla^{1,0}_\omega \partial u\right) \wedge \left(\nabla^{1,0}_\omega h \lrcorner \omega\right) \wedge e^h \omega^{n-1} \\
&= && -2\left(\nabla^{1,0}_\omega u \lrcorner \nabla^{1,0}_\omega \partial u\right) \wedge e^h \left(\nabla^{1,0}_\omega h \lrcorner \omega^n\right) \\
&= && -2\nabla^{1,0}_\omega \partial u \left(\nabla^{1,0}_\omega u, \nabla^{1,0}_\omega h\right) e^h \omega^n,
\end{aligned}$$

so we have the formula

$$\begin{aligned}
2n\,\bar\partial\Big[\left(i\nabla^{1,0}_\omega u \lrcorner \nabla^{1,0}_\omega \partial u\right) \wedge e^h \omega^{n-1}\Big] &= \operatorname{Tr}_\omega \Big[i\bar\partial\left(\nabla^{1,0}_\omega u \lrcorner \nabla^{1,0}_\omega \partial u\right)\Big] e^h \omega^n \\
&- 2\nabla^{1,0}_\omega \partial u \left(\nabla^{1,0}_\omega u, \nabla^{1,0}_\omega h\right) e^h \omega^n.
\end{aligned}$$

Then the formula (5) follows from the formula (7) and the Stokes formula. $\square$

Corollary 1 (Poincarré type inequality). *Let X be a Fano manifold of complex dimension n, let $\omega \in 2\pi c_1(X)$ be a Kähler metric and $h \in \mathcal{E}(X,\mathbb{R})$ such that $\operatorname{Ric}(\omega) - \omega = i\partial\bar\partial h$. Set $V_h := \int_X e^h \omega^n$. Then for all smooth functions $\varphi \in \mathcal{E}(X,\mathbb{R})$ we have the Poincarré type inequality*

$$\int_X |\partial\varphi|^2_\omega \, e^h \omega^n \geq \int_X \varphi^2 e^h \omega^n - \frac{1}{V_h}\left(\int_X \varphi \, e^h \omega^n\right)^2 . \tag{8}$$

Proof. Let $u \in \mathcal{E}(X,\mathbb{R})$ be an eigenfunction corresponding to the first eigenvalue $\lambda_1 > 0$ of the Laplacian $\Delta_{\omega,h}$. Then the Bochner type formula (5) *implies the inequality*

$$\lambda_1 \int_X |\partial u|^2_\omega \, e^h \omega^n \geq \int_X |\nabla_\omega u|^2_\omega \, e^h \omega^n = 2\int_X |\partial u|^2_\omega \, e^h \omega^n.$$

The fact that u can not be constant implies $\lambda_1 \geq 2$. Consider now the function $\theta := \varphi - \int_X \varphi \, e^h \omega^n / V_h$. Then the variational characterization of λ_1 implies the inequality

$$\int_X |\partial\theta|^2_\omega \, e^h \omega^n \geq \int_X \theta^2 e^h \omega^n,$$

which implies the required Poincarré type inequality (8). $\square$

4 The Kähler-Ricci flow over Fano Manifolds

Let X be a Fano manifold of complex dimension n and let $\omega \in 2\pi c_1(X)$ be a Kähler metric. Let $\mathcal{P}_\omega := \{\varphi \in \mathcal{E}(X,\mathbb{R}) \,|\, i\partial\bar\partial\varphi > -\omega\}$ be space of potentials and define $\omega_\varphi := \omega + i\partial\bar\partial\varphi$ for every $\varphi \in \mathcal{P}_\omega$. The Kähler-Ricci flow is a family of Kähler metrics $(\omega_t)_t$, solution of the evolution equation

$$\frac{d}{dt}\omega_t = \omega_t - \operatorname{Ric}(\omega_t) \tag{9}$$

with initial metric $\omega \in 2\pi c_1$. It was proved in [Cao] that the Kähler-Ricci flow $(\omega_t)_t$ exists for all $t \in [0,+\infty)$ and $(\omega_t)_t \subset 2\pi c_1$. This is because solving the equation (9) is equivalent to solve the equation in terms of potentials

$$\dot\varphi_t = \log \frac{\omega_t^n}{\omega^n} + \varphi_t + c_t - h_\omega, \tag{10}$$

where $\varphi_t \in \mathcal{P}_\omega$, $\omega_t = \omega + i\partial\bar{\partial}\varphi_t$, $\varphi_0 = 0$, $h_\omega \in \mathcal{E}(X, \mathbb{R})$ is the the real smooth function defined by the conditions $\mathrm{Ric}(\omega) = \omega + i\partial\bar{\partial}h_\omega$, $\fint_X e^{h_\omega}\omega^n = 1$ and c_t is a constant implying the normalization $\fint_X e^{-\dot{\varphi}_t}\omega_t^n = 1$. We will allways consider the Kähler-Ricci flow equation with such normalization. We remark that to find a solution $\varphi \in \mathcal{P}_\omega$ of the Einstein equation $\mathrm{Ric}(\omega_\varphi) = \omega_\varphi$, is equivalent to solve the equation

$$0 = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega.$$

This is also equivalent to the constant scalar curvature equation $\mathrm{Sc}(\omega_\varphi) = 2n$. We prove now that the evolving metrics ω_t are G-invariant if the initial metric ω is G-invariant. Let $\Box_t := \Delta_t - 2\frac{\partial}{\partial t}$. By deriving respect to a holomorphic vector field $\xi \in \mathcal{O}(T_X)(U)$ the Kähler-Ricci flow equation (10) we find

$$\Box_t(\xi.\varphi_t) + 2\xi.\varphi_t = (\mathrm{Tr}_\omega - \mathrm{Tr}_t)(L_\xi\,\omega) + 2\xi.h_\omega, \tag{11}$$

This follows from the formula

$$2\xi.\log\frac{\omega_t^n}{\omega^n} = \mathrm{Tr}_t\, L_\xi\,\omega_t - \mathrm{Tr}_\omega\, L_\xi\,\omega.$$

Let prove this formula. Set $f_t := \omega_t^n/\omega^n$. Then $L_\xi\,\omega_t^n = (\xi.f_t)\omega^n + f_t L_\xi\,\omega^n$. So we get the equalities

$$\begin{aligned}
nL_\xi\,\omega_t \wedge \omega_t^{n-1} &= (\xi.f_t)\omega^n + nf_t L_\xi\,\omega \wedge \omega^{n-1}, \\
\xi.f_t &= \frac{nL_\xi\,\omega_t \wedge \omega_t^{n-1}}{\omega^n} - f_t\frac{nL_\xi\,\omega \wedge \omega^{n-1}}{\omega^n}, \\
2\,\xi.\log\frac{\omega_t^n}{\omega^n} &= 2\,\frac{\xi.f_t}{f_t} = \mathrm{Tr}_t\, L_\xi\,\omega_t - \mathrm{Tr}_\omega\, L_\xi\,\omega,
\end{aligned}$$

which proves our formula. Let $\mathfrak{g} \subset H^0(T_X)$ be the (real) Lie algebra of G. We remark that a differential form α is G-invariant if and only if $L_\xi\,\alpha = 0$ for all $\xi \in \mathfrak{g}$. Moreover the Ricci potential h_ω of any G-invariant metric ω is also G-invariant. So by applying (11) with $\xi \in \mathfrak{g}$ we find that the function $v_t := \xi.\varphi_t$ is solution of the equation $\Box_t v_t = -2v_t$ with initial data $v_0 = 0$. By uniqueness of the solutions we get $v_t = 0$ and so the potential φ_t is G-invariant for all times t.

Kähler-Ricci solitons.

Let ω be a Kähler metric and $u \in \mathcal{E}(X, \mathbb{R})$ be a smooth real valued function. Then $\nabla_\omega u \lrcorner\, \omega = -du \cdot J = -i\partial u + i\bar{\partial}u$ and $L_{\nabla_\omega u}\,\omega = d(\nabla_\omega u \lrcorner\, \omega) = 2i\partial\bar{\partial}u$. Let now X be a Fano manifold and $\omega \in 2\pi c_1$ be a Kähler metric. Then

$$\omega - \mathrm{Ric}(\omega) = 2i\partial\bar{\partial}u = L_{\nabla_\omega u}\,\omega.$$

If $\nabla_\omega u \in \mathcal{O}(T_{X,J})(X)$ then ω is called a Kähler-Ricci soliton. We remind that $\nabla_\omega^{1,0}\partial u = 0$ if and only if the vector field $\nabla_\omega u$ is holomorphic. So $\omega \in 2\pi c_1$ is a Kähler-Ricci soliton if and only if the Ricci potential $u \in \mathcal{E}(X, \mathbb{R})$, $\omega - \mathrm{Ric}(\omega) = 2i\partial\bar{\partial}u$ satisfies the equation $\nabla_\omega^{1,0}\partial u = 0$. Let $(\Phi_t)_{t\in\mathbb{R}}$ be the 1-parameter Group of holomorphic automorphisms of X induced by $\nabla_\omega u \in \mathcal{O}(T_X)(X)$. Let $\omega_t := \Phi_t^*\omega = \omega + i\partial\bar{\partial}\varphi_t$ and $u_t = u \circ \Phi_t$. Then we get the Kähler-Ricci flow equation

$$\frac{d}{dt}\omega_t = \omega_t - \mathrm{Ric}(\omega_t) = 2i\partial\bar{\partial}u_t, \quad \text{with} \quad \nabla_t^{1,0}\partial u_t = 0.$$

Remark 1. If the Futaki invariant $f_{2\pi c_1}$ is zero then all Kähler-Ricci solitons are Kähler-Einstein metrics. In fact by definition of the Futaki invariant

$$f_{2\pi c_1}(\nabla_\omega u) = -2 \fint_X \nabla_\omega u \,.u\, \omega^n = -2 \fint_X |\nabla_\omega u|^2_\omega \,\omega^n.$$

Moreover this formula shows that the existence of a Kähler-Ricci solitons with $\nabla_\omega u \neq 0$ implies the non existence of Kähler-Einstein metrics.

Remark 2. Consider again a smooth real valued function $u \in \mathcal{E}(X, \mathbb{R})$. Then $L_{J\nabla_\omega u}\,\omega = d(J\nabla_\omega u \,\lrcorner\, \omega) = 0$ since $J\nabla_\omega u \,\lrcorner\, \omega = -\omega(\nabla_\omega u, J\cdot) = -du$. Let now X be a Fano manifold. For any Kähler metric ω and any smooth real vector field $\xi \in \mathcal{E}(T_X)(X)$ such that $L_{J\xi}\,\omega = 0$ there exist a smooth real valued function $u \in \mathcal{E}(X, \mathbb{R})$ such that $\xi = \nabla_\omega u$. In fact consider the decomposition $\xi = \xi' + \xi''$, with $\xi'' = \overline{\xi'} \in \mathcal{E}(T^{0,1}_{X,J})(X)$. Then

$$0 = L_{J\xi}\,\omega = d(J\xi \,\lrcorner\, \omega) = id(\xi' \,\lrcorner\, \omega) - id(\xi'' \,\lrcorner\, \omega) = i\partial(\xi' \,\lrcorner\, \omega) - i\bar{\partial}(\xi'' \,\lrcorner\, \omega),$$

since $\bar{\partial}(\xi' \,\lrcorner\, \omega) = 0$ and $\partial(\xi'' \,\lrcorner\, \omega) = 0$ by decomposition of the degree. We deduce also the equality $\partial(\xi' \,\lrcorner\, \omega) = \bar{\partial}(\xi'' \,\lrcorner\, \omega)$. The fact that X is Fano and the equality $\bar{\partial}(\xi' \,\lrcorner\, \omega) = 0$ implient the existence of $u \in \mathcal{E}(X, \mathbb{C})$ such that $\xi' \,\lrcorner\, \omega = i\bar{\partial}u$ and by conjugation $\xi'' \,\lrcorner\, \omega = -i\partial\bar{u}$. Then the equality $\partial(\xi' \,\lrcorner\, \omega) = \bar{\partial}(\xi'' \,\lrcorner\, \omega)$ implies $i\partial\bar{\partial}(u - \bar{u}) = 0$, which means that the function u can be chosen with real values.

Remark 3. Let (X, ω) be a compact Kähler manifold such that $\omega - \mathrm{Ric}(\omega) = L_\xi\,\omega$, for some smooth real vector field $\xi \in \mathcal{E}(T_X)(X)$. So $L_\xi\,\omega = d(\xi \,\lrcorner\, \omega)$ is a real d-exact $(1,1)$-form. By Hodge Theory there exist $u \in \mathcal{E}(X, \mathbb{R})$ such that $L_\xi\,\omega = i\partial\bar{\partial}u$. So we deduce that X is a Fano manifold and $\omega \in 2\pi c_1$. Using remark 2 we find that a Kähler metric ω over a compact Kähler manifold X is a Kähler-Ricci soliton if and only if there exist a real holomorphic vector field $\xi \in \mathcal{O}(T_X)(X)$ such that $\omega - \mathrm{Ric}(\omega) = L_\xi\,\omega$ and $L_{J\xi}\,\omega = 0$. Moreover the holomorphic vector field ξ is uniquely determined by the metric ω, since it is uniquely determined by the Ricci potential of ω.

Perelman's uniform estimates for the Kähler-Ricci flow.
We have the following fundamental result due to Perelman.

Theorem 2 (Perelman) *Over a Fano manifold X of complex dimension n, the Kähler-Ricci flow $\frac{d}{dt}\omega_t = \omega_t - \mathrm{Ric}_t = i\partial\bar{\partial}\dot{\varphi}_t$ satisfies the uniform estimates $|\dot{\varphi}_t|$, $|\nabla_t\,\dot{\varphi}_t|_t$, $|\Delta_t\dot{\varphi}_t|$, $\mathrm{Diam}_t(X)$, $\mathrm{Sc}_t \leq C$, where the Ricci potential $\dot{\varphi}_t$ is normalized by the condition $\fint_X e^{-\dot{\varphi}_t}\omega_t^n = 1$.*

Set $u_t := \dot{\varphi}_t$, $a_t := \dot{c}_t$. Then time deriving the Kähler-Ricci flow equation (10), we find the identity

$$\Box_t u_t = -2u_t - 2a_t. \tag{12}$$

The first step in proving Perelman's theorem consist in showing the boundedness of the constant a_t. We give a proof here. By deriving the integral normalization

$\fint_X e^{-u_t}\omega_t^n = 1$ and using the equation (12), we find the equalities

$$\begin{aligned} 0 = \frac{d}{dt}\fint_X e^{-u_t}\omega_t^n &= -\fint_X \dot{u}_t\, e^{-u_t}\omega_t^n + 2^{-1}\fint_X \Delta_t u_t\, e^{-u_t}\omega_t^n \\ &= -a_t\fint_X e^{-u_t}\omega_t^n - \fint_X u_t\, e^{-u_t}\omega_t^n , \end{aligned}$$

which implient

$$a_t = -\fint_X u_t\, e^{-u_t}\omega_t^n \geq -\fint_X (u_t)_+\, e^{-u_t}\omega_t^n ,$$

where $(u_t)_+ := \max\{u_t, 0\}$. The fact that the function $f(x) := -xe^{-x}$ is bounded over the interval $[0, +\infty)$ implies the uniform estimate $a_t \geq -C$. We prove now the upper bound of a_t. Perelman show this by using the monotonicity of his μ functional along the Kähler-Ricci flow. We realize that the the upper bound of a_t follows in a classical way by using the generalized Bochner-Kodaira Formula. In fact using the Kähler-Ricci flow identity (12) and the identity $\Delta_t e^{-u_t} = (2|\partial u_t|_t^2 - \Delta_t u_t)e^{-u_t}$, we find

$$\begin{aligned} -\dot{a}_t &= \fint_X \Big[\dot{u}_t - u_t\,(\dot{u}_t - \Delta_t u_t/2)\Big] e^{-u_t}\omega_t^n = \fint_X \Big[\dot{u}_t - u_t\,(u_t + a_t)\Big] e^{-u_t}\omega_t^n \\ &= \frac{1}{2}\fint_X \Delta_t u_t e^{-u_t}\omega_t^n + \fint_X u_t e^{-u_t}\omega_t^n + a_t - \fint_X u_t^2 e^{-u_t}\omega_t^n + a_t^2 \\ &= \fint_X |\partial u_t|_t^2 e^{-u_t}\omega_t^n - \fint_X u_t^2 e^{-u_t}\omega_t^n + a_t^2 \end{aligned}$$

By the Poincaré type inequality 8 in the Fano case we deduce $-\dot{a}_t \geq 0$. This implies the upper bound of the normalizing constants a_t. Consider now Perelman's functional

$$\mathcal{W}(\omega, f, \tau) := (4\pi c_1 \tau)^{-n}\int_X \Big[\tau\left(|\nabla_\omega f|_\omega^2 + \mathrm{Sc}_\omega\right) + f - 2n\Big] e^{-f}\omega^n .$$

Using the identity $\Delta_t u_t = 2n - \mathrm{Sc}_t$ we get

$$\begin{aligned} \mathcal{W}_t := \mathcal{W}(\omega_t, u_t, 1/2) &= \fint_X \left[\frac{1}{2}\left(|\nabla_t\, u_t|_t^2 + \mathrm{Sc}_t\right) + u_t - 2n\right] e^{-u_t}\omega_t^n \\ &= \fint_X \left[\frac{1}{2}\left(|\nabla_t\, u_t|_t^2 - \Delta_t u_t\right) + u_t\right] e^{-u_t}\omega_t^n - n \\ &= \fint_X \left(\frac{1}{2}\Delta_t e^{-u_t} + u_t e^{-u_t}\right)\omega_t^n - n \\ &= -a_t - n\,. \end{aligned}$$

So we find the inequality $\dot{\mathcal{W}}_t = -\dot{a}_t \geq 0$. Suppose now that $\dot{\mathcal{W}}_t = 0$ for some time t. Then we get the equality case in the Poincaré inequality

$$\int_X |\nabla_t\, \theta_t|_t^2\, e^{-u_t}\omega_t^n = 2\int_X \theta_t^2\, e^{-u_t}\omega_t^n\,,$$

with $\theta_t := u_t - \fint_X u_t e^{-u_t}\omega_t^n = u_t + a_t$. The variational characterization of the first non zero eigenvalue $\lambda_1(\hat{\Delta}_t)$ of the generalized Laplacian $\hat{\Delta}_t := \Delta_{t,-u_t}$ implies $2 \geq \lambda_1(\hat{\Delta}_t)$. By the other hand the generalized Bochner-Kodaira formula implies $\lambda_1(\hat{\Delta}_t) \geq 2$. So $\lambda_1(\hat{\Delta}_t) = 2$ and $\hat{\Delta}_t u_t + 2(u_t + a_t) = 0$. By plugging this in to the generalized Bochner-Kodaira formula we get

$$\int\limits_X |\bar{\partial}\,\nabla_t^{1,0} u_t|_t^2\, e^{-u_t}\omega_t^n = -\int\limits_X \Big\langle \partial\hat{\Delta}_t u_t, \partial u \Big\rangle_t e^{-u_t}\omega_t^n - \int\limits_X |\nabla_t\, u_t|_t^2\, e^{-u_t}\omega_t^n = 0\,.$$

So ω_t is a Kähler-Ricci soliton and this will hold for all times. We have prove in conclusion the proposition 1.1.

The generalized functionals by Aubin.
The generalized functionals $I_\omega,\, J_\omega : \mathcal{P}_\omega \to [0, +\infty)$ by Aubin, [Aub1] are defined by the formulas

$$\begin{aligned}
I_\omega(\varphi) \quad &:= \quad \fint\limits_X \varphi\left(\omega^n - \omega_\varphi^n\right) = \sum_{k=0}^{n-1} \fint\limits_X i\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^k \wedge \omega_\varphi^{n-k-1} \\
J_\omega(\varphi) \quad &:= \quad \sum_{k=0}^{n-1} \frac{k+1}{n+1} \fint\limits_X i\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^k \wedge \omega_\varphi^{n-k-1} \\
&= \quad \fint\limits_X \varphi\,\omega^n - \frac{1}{n+1}\sum_{k=0}^{n} \fint\limits_X \varphi\,\omega^k \wedge \omega_\varphi^{n-k}.
\end{aligned}$$

We have the obvious inequalities, $0 \leq I_\omega \leq (n+1)J_\omega$.

The K-energy functional of the anticanonical class $2\pi c_1$. We remind that the Einstein equation $\mathrm{Ric}(\omega_\varphi) = \omega_\varphi$, is equivalent to the constant scalar curvature equation $\mathrm{Sc}(\omega_\varphi) = 2n$. This last equation is the Euler-Lagrange equation of Mabuchi's [Mab] K-energy functional $\nu_\omega : \mathcal{P}_\omega \to \mathbb{R}$

$$\nu_\omega(\varphi) := \fint\limits_X \left(\log\frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega\right)\omega_\varphi^n - \frac{1}{n+1}\sum_{k=0}^{n} \fint\limits_X \varphi\,\omega^k \wedge \omega_\varphi^{n-k} + \fint\limits_X h_\omega \omega^n.$$

In fact for every $\mathcal{C}^\infty$ path $(\varphi_t)_{t\in(-\varepsilon,\varepsilon)} \subset \mathcal{P}_\omega$ we have the identity

$$\frac{d}{dt}\nu_\omega(\varphi_t) = -\frac{1}{2}\fint\limits_X \dot{\varphi}_t\Big(\mathrm{Sc}(\omega_t) - 2n\Big)\omega_t^n\,, \tag{13}$$

where $\dot{\varphi}_t := \frac{\partial}{\partial t}\varphi_t$ and $\omega_t := \omega_{\varphi_t}$. We remark that under the Kähler-Ricci flow we have the identity $\mathrm{Sc}(\omega_t) = 2n - \Delta_{\omega_t}\dot{\varphi}_t$. Then using the identity (13) we deduce the inequality

$$\frac{d}{dt}\nu_\omega(\varphi_t) = 2^{-1}\fint\limits_X \dot{\varphi}_t \Delta_{\omega_t}\dot{\varphi}_t\,\omega_t^n = -n\fint\limits_X i\partial\dot{\varphi}_t \wedge \bar{\partial}\dot{\varphi}_t \wedge \omega_t^{n-1} \leq 0,$$

which shows that the K-energy decreases under the Kähler-Ricci flow. We remind also the following Tian's [Tia] fundamental result.

Theorem 3 (Tian's G-properness) *Let X be a Fano manifold admitting a G-invariant Kähler-Einstein metric $\hat{\omega} \in 2\pi c_1$. Then there exists two constants $\delta > 0$, $C > 0$ such that the inequality $\nu_{\hat{\omega}}(\varphi) \geq J_{\hat{\omega}}(\varphi)^{\delta} - C$ hold for all G-invariant potentials $\varphi \in \mathcal{P}_{\hat{\omega}}$.*

By using the cocycle condition we deduce that for all G-invariant Kähler metrics $\omega \in 2\pi c_1$ there exist an increasing function $\mu : \mathbb{R} \rightarrow [c, +\infty)$, with $\lim_{t \rightarrow +\infty} \mu(t) = +\infty$ such that $\nu_{\omega}(\varphi) \geq \mu(J_{\omega}(\varphi))$ for all G-invariant potentials $\varphi \in \mathcal{P}_{\omega}$.

5 Tian-Zhu's C^0-uniform estimate

We start by proving the following elementary lemma

Lemma 5 *Let (X, ω) be a polarized Fano manifold with $\omega \in 2\pi c_1$, let $(\omega_t)_t$ be a Kähler-Ricci flow and let and G be a compact maximal subgroup of the identity component of the group of automorphisms of X.*
A). *Suppose there exist a constant $k > 0$ such that $\omega_t^n \geq k\omega^n$ for all times $t \geq 0$. Then the Aubin's Functional J_{ω} is uniformly bounded along this Kähler-Ricci flow.*
B). *Suppose X admits a G-invariant Kähler-Einstein metric. Then the Kähler-Ricci flow with G-invariant initial metric ω satisfies the uniform estimate $\omega_t^n \geq k\,\omega^n$, $k > 0$ for all times $t \geq 0$.*

So part B of this lemma prove one implication in theorem 1.

Proof. Set $\hat{\varphi}_t := \varphi_t + c_t$. By writing the Kähler-Ricci flow equation under the form

$$\hat{\varphi}_t = \dot{\varphi}_t - \log \frac{\omega_t^n}{\omega^n} + h_{\omega}\,,$$

using the Perelman's uniform estimate $|\dot{\varphi}_t| \leq C$ and the inequality $\omega_t^n \geq k\omega^n$ we find the uniform estimate $\hat{\varphi}_t \leq C - \log k + h_{\omega}$. Reminding the expression of the K-energy functional we deduce the identity along the Kähler-Ricci flow

$$\nu_{\omega}(\varphi_t) = \fint_X \dot{\varphi}_t\,\omega_t^n + J_{\omega}(\varphi_t) - \fint_X \hat{\varphi}_t\,\omega^n + \fint_X h_{\omega}\,\omega^n. \tag{14}$$

Then using; the fact that the K-energy functional is nonincreasing along the Kähler-Ricci flow, the inequality $-\fint_X \dot{\varphi}_t\,\omega_t^n \leq 0$ (which follows from the integral normalization of $\dot{\varphi}_t$) and the previous estimate $\hat{\varphi}_t \leq C$, we deduce the uniform estimate

$$0 \leq J_{\omega}(\varphi_t) = \nu_{\omega}(\varphi_t) - \fint_X \dot{\varphi}_t\,\omega_t^n + \fint_X \hat{\varphi}_t\,\omega^n - \fint_X h_{\omega}\,\omega^n \leq \nu_{\omega}(\varphi_0) + C.$$

We prove now part B. The existence of a G-invariant Kähler-Einstein metric implies the G-properness of the K-energy functional. Then using the fact that the K-energy functional is nonincreasing along the Kähler-Ricci flow we deduce that the energy functional J_{ω} is bounded along the Kähler-Ricci flow $(\omega_t)_t$ with G-invariant initial metric ω. Then the identity (14) combined with the fact that the K-energy functional is bounded from below implies the uniform

estimate $\fint_X \hat{\varphi}_t\, \omega^n \leq C$. By the properties of the Green function we deduce the inequality

$$\hat{\varphi}_t \leq \fint_X \hat{\varphi}_t\, \omega^n + C' \leq C\,.$$

This uniform estimate is equivalent to the uniform estimate $\omega_t^n \geq k\, \omega^n$ by means of the Kähler-Ricci flow equation and Perelman's uniform estimate $|\dot{\varphi}_t| \leq C$. $\square$
We remind now the following result [Ti-Zh].

Proposition 5.1 *Let (X, ω) be a compact Kähler manifold of complex dimension n, let $\varphi \in \mathcal{P}_\omega$ and $f := \omega_\varphi^n / \omega^n$. Then for all $\varepsilon \in (0, \varepsilon_0]$, $\delta \in (0, \delta_0]$ there exists constants $C, C' > 0$ depending only on ω, ε_0, δ_0 such that*

$$\mathrm{Osc}(\varphi) \leq C \left(\frac{1}{\varepsilon \delta}\right)^{n+\delta} \|f\|^\delta_{L^{1+\varepsilon}(X,\omega)} + C'\,.$$

Proposition 5.2 *Let (X, ω) be a polarized Fano manifold with $\omega \in 2\pi c_1$ and let $(\omega_t)_t$ be a Kähler-Ricci flow admiting a constant $k > 0$ such that $\omega_t^n \geq k\omega^n$ for all times $t \geq 0$. Then this Kähler-Ricci flow satisfies the uniform estimate $|\varphi_t + c_t| \leq K_0$, for some constant $K_0 > 0$ independent of $t \geq 0$.*

Proof. The argument here is the same as in [Ti-Zh]. We start proving the uniform estimate $|\max_X \hat{\varphi}_t| \leq C$ for all $t \geq 0$. We first remark that if a real function u satisfies the integral equality $\int_X (e^{-u} - 1)\omega^n = 0$ then $\max_X u \geq 0$. In fact if not $0 > \max_X u \geq u$ and this implies $e^{-u} > 1$, which contradict the integral equality. By definition of Kähler-Ricci flow we have the integral identity

$$\int_X e^{h - \hat{\varphi}_t} \omega^n = \int_X e^{-\dot{\varphi}_t} \omega_t^n = \int_X \omega^n, \tag{15}$$

for all $t \in [0, +\infty)$. Then applying the previous remark with $u := \hat{\varphi}_t - h$, we find the inequality $\max_X (\hat{\varphi}_t - h) \geq 0$, which gives the estimate $\max_X \hat{\varphi}_t \geq -C$. Moreover the argument in the proof of A of lemma 5 implies $\hat{\varphi}_t \leq C$. The equality (15) implies that the function $h - \hat{\varphi}_t$ change signs and so we get $\|h - \hat{\varphi}_t\|_{C^0(X)} \leq \mathrm{Osc}(h - \hat{\varphi}_t)$, which implies $\|\hat{\varphi}_t\|_{C^0(X)} \leq \mathrm{Osc}(\hat{\varphi}_t) + C$. By proposition 5.1 we need to prove a uniform bound for the integral $\int_X e^{-(1+\varepsilon)\hat{\varphi}_t} \omega^n$ for some $\varepsilon > 0$. Set $\theta_t := \max_X \varphi_t - \varphi_t \geq 0$. Then

$$e^{-(1+\varepsilon)\hat{\varphi}_t} \omega^n = e^{\varepsilon \theta_t - \varepsilon \max \hat{\varphi}_t - \hat{\varphi}_t} \omega^n \leq C e^{\varepsilon \theta_t - \hat{\varphi}_t} \omega^n \leq C' e^{\varepsilon \theta_t} \omega_t^n\,.$$

The last inequality follows from the Kähler-Ricci flow equation and Perelman's uniform estimate $|\dot{\varphi}_t| \leq C$. So it is sufficient to prove an uniform bound for the integral $\int_X e^{\varepsilon \theta_t} \omega_t^n$. In order to prove this we consider the classic inequality

$$0 \leq I_\omega(\varphi_t) = \fint_X \hat{\varphi}_t\, (\omega^n - \omega_t^n) \leq (n+1) J_\omega(\varphi_t) \leq C\,,$$

where $C > 0$ is the uniform constant provided by lemma 5. We deduce

$$-\fint_X \hat{\varphi}_t\, \omega_t^n \leq C - \fint_X \hat{\varphi}_t\, \omega^n \leq 2C.$$

The last inequality follows from the estimate $\fint_X e^{-\hat{\varphi}_t}\,\omega^n \leq C$ that we get from the identity (15). So we have obtain the uniform estimate

$$0 \leq \fint_X \theta_t\,\omega_t^n \leq C \tag{16}$$

For all integers $p \geq 1$ we have the equalities

$$\begin{aligned}
\int_X \theta_t^p\left(\omega_t^n - \omega_t^{n-1}\wedge\omega\right) &= -\int_X \theta_t^p\, i\partial\bar{\partial}\theta_t \wedge \omega_t^{n-1}\\
&= p\int_X \theta_t^{p-1} i\partial\theta_t \wedge \bar{\partial}\theta_t \wedge \omega_t^{n-1}\\
&= p\int_X \theta_t^{\frac{p-1}{2}} i\partial\theta_t \wedge \theta_t^{\frac{p-1}{2}}\bar{\partial}\theta_t \wedge \omega_t^{n-1}\\
&= \frac{4p}{(p+1)^2}\int_X i\partial\theta_t^{\frac{p+1}{2}} \wedge \bar{\partial}\theta_t^{\frac{p+1}{2}} \wedge \omega_t^{n-1}\\
&= \frac{4p}{n(p+1)^2}\int_X |\partial\theta_t^{\frac{p+1}{2}}|_t^2\,\omega_t^n\,.
\end{aligned}$$

This implies the inequality

$$\int_X |\partial\theta_t^{\frac{p+1}{2}}|_t^2\,\omega_t^n \leq \frac{n(p+1)^2}{4p}\int_X \theta_t^p\,\omega_t^n. \tag{17}$$

Remember now the Kähler-Ricci flow identity $i\partial\bar{\partial}\dot{\varphi}_t = \omega_t - \mathrm{Ric}(\omega_t)$. Then as in [Ti-Zh] by applying the Poincarré type inequality (corollary 8) to the function $\theta_t^{\frac{p+1}{2}}$, with metric ω_t and $h = -\dot{\varphi}_t$, we deduce

$$\int_X |\partial\theta_t^{\frac{p+1}{2}}|_t^2\, e^{-\dot{\varphi}_t}\omega_t^n \geq \int_X \theta_t^{p+1} e^{-\dot{\varphi}_t}\omega_t^n - (2\pi c_1)^{-n}\left(\int_X \theta_t^{\frac{p+1}{2}}\, e^{-\dot{\varphi}_t}\omega_t^n\right)^2.$$

By applying the Hölder inequality to the last therm, using Perelman uniform estimate $|\dot{\varphi}_t| \leq C$ and the inequality (17) we deduce

$$\int_X \theta_t^{p+1} e^{-\dot{\varphi}_t}\omega_t^n \leq Cp\int_X \theta_t^p\,\omega_t^n + C\int_X \theta_t^p\, e^{-\dot{\varphi}_t}\omega_t^n \cdot \int_X \theta_t\, e^{-\dot{\varphi}_t}\omega_t^n.$$

Using again the estimate $|\dot{\varphi}_t| \leq C$ and the estimate (16) we find

$$\int_X \theta_t^{p+1}\,\omega_t^n \leq C(p+1)\int_X \theta_t^p\,\omega_t^n.$$

By iteration $\int_X \theta_t^p\,\omega_t^n \leq C^p p!$. Thus

$$\int_X e^{\varepsilon\theta_t}\omega_t^n = \sum_{p=0}^{\infty}\frac{\varepsilon^p}{p!}\int_X \theta_t^p\,\omega_t^n \leq \sum_{p=0}^{\infty}(\varepsilon C)^p\,.$$

So we choose $0 < \varepsilon < 1/C$. $\square$

6 The Yau's C^2 and Calabi's C^3 uniform estimates for the Kähler-Ricci flow

We start with some notations and definitions. Let (X,ω) be a Kähler manifold of complex dimension n. Consider the function $\lambda_1^\omega : X \to \mathbb{R}$ defined by the formula

$$\lambda_1^\omega(x) := \min_{\xi \in T_{X,x}^{\otimes 2} \smallsetminus 0_x} \mathcal{C}_{X,J}^\omega(\xi,\xi)|\xi|_\omega^{-2}.$$

So $\lambda_1^\omega(x)$ is the smallest eigenvalue of the Chern Curvature form $\mathcal{C}_{X,J}^\omega(x)$. It is well known (see [Kat], chap II, sec 5.1, theorem 5.1, pag 107) that the function λ_1^ω is continuous. In order to simplify the notations we will use Einstein convention on sums. Moreover we will note by Tr_φ the trace operator corresponding to the metric ω_φ. With such notations we have the following proposition which is obtained by some computations in [Yau].

Proposition 6.1 *Let (X,ω) be a Kähler manifold of complex dimension n. Then for every potential $\varphi \in \mathcal{P}_\omega$ for the Kähler metric ω we have the intrinsic inequality*

$$2\,\mathrm{Tr}_\omega \mathrm{Ric}(\omega_\varphi) \geq -\Delta_\varphi \Delta_\omega \varphi + 4\lambda_1^\omega\,(2n + \Delta_\omega\varphi)\,\mathrm{Tr}_\varphi\,\omega + \frac{2|\partial\Delta_\omega\varphi|_\varphi^2}{2n+\Delta_\omega\varphi}.$$

Proof. Let $(z_1,...,z_n)$ be ω-geodesic holomorphic coordinates with center a point x such that the metric ω_φ can be written in diagonal form in x. Explicitly $\omega = \frac{i}{2}\omega_{l,\bar r}\, dz_l \wedge d\bar z_r$ and $\omega_\varphi = \frac{i}{2}(\omega_\varphi)_{l,\bar r}\, dz_l \wedge d\bar z_r$, with

$$\begin{gathered}
\omega_{l,\bar r} = \delta_{l,r} - C_{r,l}^{j,\bar k} z_j \bar z_k + O(|z|^3),\\
\mathcal{C}_\omega(T_{X,J})(x) = C_{l,r}^{j,\bar k}\,(dz_j \wedge d\bar z_k) \otimes dz_r \otimes_J \frac{\partial}{\partial z_l},\\
\mathcal{C}_{X,J}^\omega(x) = C_{r,l}^{j,\bar k}\, dz_j \otimes dz_l \otimes d\bar z_k \otimes d\bar z_r,\\
\overline{C_{l,r}^{j,\bar k}} = C_{r,l}^{k,\bar j},\; C_{l,r}^{j,\bar k} = C_{l,j}^{r,\bar k} = C_{k,r}^{j,\bar l},\\
(\omega_\varphi)_{l,\bar r} = \delta_{l,r} + 2\varphi_{l,\bar r} + O(|z|), \quad 2\varphi_{l,\bar r}(0) = 2\delta_{l,r}\varphi_{l,\bar l}(0) > -1,
\end{gathered}$$

where $\varphi_{l,\bar r} := \frac{\partial^2\varphi}{\partial z_l \partial \bar z_r}$. In particular we deduce the following expressions for the inverse matrixs

$$\omega^{l,\bar r} = \delta_{l,r} + C_{r,l}^{j,\bar k} z_j \bar z_k + O(|z|^3), \qquad (\omega_\varphi)^{l,\bar r} = \frac{\delta_{l,r}}{1+2\varphi_{l,\bar l}} + O(|z|).$$

Moreover we deduce the local expressions

$$\begin{gathered}
\Delta_\omega\varphi = 4\omega^{l,\bar r}\varphi_{r,\bar l} = 4(\varphi_{l,\bar l} + C_{r,l}^{j,\bar k}\varphi_{r,\bar l}\, z_j \bar z_k) + O(|z|^3),\\
\Delta_\varphi u = 4(\omega_\varphi)^{l,\bar r} u_{r,\bar l} = \frac{4u_{l,\bar l}}{1+2\varphi_{l,\bar l}} + O(|z|)
\end{gathered}$$

for every smooth function u. Using this two expressions we find the equality at the point x

$$\Delta_\varphi\Delta_\omega\varphi = \frac{4^2}{1+2\varphi_{l,\bar l}}(\varphi_{l,\bar l,k,\bar k} + C_{r,k}^{l,\bar l}\varphi_{r,\bar k}) = \frac{4^2}{1+2\varphi_{l,\bar l}}(\varphi_{l,\bar l,k,\bar k} + C_{k,k}^{l,\bar l}\varphi_{k,\bar k}). \quad (18)$$

Using the expression (2) of the coefficients of the curvature form respect to the complex frame $(\zeta_k) := (\partial/\partial z_k)$, we find the following local expression for the Ricci tensor

$$\mathrm{Ric}(\omega) = -\Big(i\partial\bar{\partial}\omega_{l,\bar{r}} - i\partial\omega_{l,\bar{s}}\,\omega^{s,\bar{t}} \wedge \bar{\partial}\omega_{t,\bar{r}}\Big)\omega^{r,\bar{l}}.$$

All the computations that will follow are refereed to the point x. Expanding the analogue expression for $\mathrm{Ric}(\omega_\varphi)$ we get the equality.

$$\mathrm{Ric}(\omega_\varphi) = \Big(iC_{r,l} - 2i\partial\bar{\partial}\varphi_{l,\bar{r}} + 4i\partial\varphi_{l,\bar{s}}\,\omega_\varphi^{s,\bar{t}} \wedge \bar{\partial}\varphi_{t,\bar{r}}\Big)\omega_\varphi^{r,\bar{l}}.$$

Taking the trace respect to ω of the Ricci tensor $\mathrm{Ric}(\omega_\varphi)$ we find the expression

$$\begin{aligned}
\mathrm{Tr}_\omega \mathrm{Ric}(\omega_\varphi) &= 4\Big(C_{r,l}^{k,\bar{k}} - 2\varphi_{k,\bar{k},l,\bar{r}} + 4\varphi_{k,l,\bar{s}}\,\omega_\varphi^{s,\bar{t}}\,\varphi_{\bar{k},t,\bar{r}}\Big)\omega_\varphi^{r,\bar{l}} \\
&= 4\Big(C_{l,l}^{k,\bar{k}} - 2\varphi_{k,\bar{k},l,\bar{l}} + \frac{4\varphi_{k,l,\bar{s}}\,\varphi_{\bar{k},s,\bar{l}}}{1+2\varphi_{s,\bar{s}}}\Big)\frac{1}{1+2\varphi_{l,\bar{l}}}.
\end{aligned}$$

Using the symmetry $C_{l,l}^{k,\bar{k}} = C_{k,k}^{l,\bar{l}} \in \mathbb{R}$ and the identity (18) we find the equality

$$\mathrm{Tr}_\omega \mathrm{Ric}(\omega_\varphi) = -\frac{1}{2}\Delta_\varphi\Delta_\omega\varphi + 4C_{k,k}^{l,\bar{l}}\frac{1+2\varphi_{k,\bar{k}}}{1+2\varphi_{l,\bar{l}}} + \frac{16\varphi_{k,l,\bar{s}}\,\varphi_{\bar{k},s,\bar{l}}}{(1+2\varphi_{s,\bar{s}})(1+2\varphi_{l,\bar{l}})}. \qquad (19)$$

The inequality $C_{k,k}^{l,\bar{l}} = \mathcal{C}_{X,J}^{\omega}(\frac{\partial}{\partial x_l} \otimes \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \otimes \frac{\partial}{\partial x_k})(x) \geq \lambda_1^\omega(x)$, implies the inequality

$$4C_{k,k}^{l,\bar{l}}\frac{1+2\varphi_{k,\bar{k}}}{1+2\varphi_{l,\bar{l}}} \geq 2\lambda_1^\omega\,(2n+\Delta_\omega\varphi)\,\mathrm{Tr}_\varphi\,\omega\,(x).$$

Then the conclusion of the proof of the proposition will follows from the inequality

$$\frac{16\varphi_{k,l,\bar{s}}\,\varphi_{\bar{k},s,\bar{l}}}{(1+2\varphi_{s,\bar{s}})(1+2\varphi_{l,\bar{l}})} \geq \frac{|\partial\Delta_\omega\varphi|_\varphi^2}{2n+\Delta_\omega\varphi}. \qquad (20)$$

Let prove this inequality. We have

$$\begin{aligned}
|\partial\Delta_\omega\varphi|_\varphi^2 &= 2\sum_{k,l}\omega_\varphi^{k,\bar{l}}\,\partial_l\Delta_\omega\varphi\,\partial_{\bar{k}}\Delta_\omega\varphi = \sum_{j,k,l}\frac{2\cdot 4^2\varphi_{l,j,\bar{j}}\,\varphi_{\bar{l},k,\bar{k}}}{1+2\varphi_{l,\bar{l}}} \\
&= \sum_l \frac{2\cdot 4^2}{1+2\varphi_{l,\bar{l}}}\left|\sum_j \frac{\varphi_{l,j,\bar{j}}}{\sqrt{1+2\varphi_{j,\bar{j}}}}\sqrt{1+2\varphi_{j,\bar{j}}}\right|^2.
\end{aligned}$$

Applying the Cauchy-Schwartz inequality to the norm, we find the inequality

$$\begin{aligned}
|\partial\Delta_\omega\varphi|_\varphi^2 &\leq \sum_l \frac{2\cdot 4^2}{1+2\varphi_{l,\bar{l}}}\left(\sum_j \frac{|\varphi_{l,j,\bar{j}}|^2}{1+2\varphi_{j,\bar{j}}}\right)\left(\sum_k (1+2\varphi_{k,\bar{k}})\right) \\
&= (2n+\Delta_\omega\varphi)\sum_{j,l}\frac{16|\varphi_{l,j,\bar{j}}|^2}{(1+2\varphi_{j,\bar{j}})(1+2\varphi_{l,\bar{l}})} \\
&\leq (2n+\Delta_\omega\varphi)\sum_{k,l,j}\frac{16\varphi_{k,l,\bar{j}}\,\varphi_{\bar{k},j,\bar{l}}}{(1+2\varphi_{j,\bar{j}})(1+2\varphi_{l,\bar{l}})},
\end{aligned}$$

which conclude the proof of the inequality (20). □

We will note by ω^* and ω^*_φ the corresponding dual elements of ω and ω_φ. Let h^* and h^*_φ the corresponding hermitian metrics over the complex vector bundle $T^*_{X,J}$. In local complex coordinates we have the expressions $\omega^* = 2i\,\omega^{l\bar{k}}\frac{\partial}{\partial z_k}\wedge\frac{\partial}{\partial \bar{z}_l}$ and $h^* = 4\,\omega^{l\bar{k}}\frac{\partial}{\partial z_k}\otimes\frac{\partial}{\partial \bar{z}_l}$. We remind also that if (V,J) is a complex vector space equipped with a hermitian metric h then the corresponding hermitian metric $h_{\mathbb{C}}$ over the complexified vector space $(V\otimes_{\mathbb{R}}\mathbb{C}, i)$ is defined by the formula

$$2h_{\mathbb{C}}(v,w) := h(v,\overline{w}) + \overline{h(\overline{v},w)}, \quad v,w\in V\otimes_{\mathbb{R}}\mathbb{C},$$

where we still note by h the $\mathbb{C}$-linear extension of h. Consider now the complex vector bundles $F := \Lambda^{1,0}_J T^*_X$, $E := F^{\otimes 2}\otimes\overline{F}$ and the hermitian vector bundles

$$\begin{array}{rcl}
(E,\langle\cdot,\cdot\rangle_\omega) & := & (F,h^*)\otimes(F,h^*)\otimes(\overline{F},h^*)\\
(E,\langle\cdot,\cdot\rangle_\varphi) & := & (F,h^*_\varphi)\otimes(F,h^*_\varphi)\otimes(\overline{F},h^*_\varphi)\\
(E,\langle\cdot,\cdot\rangle_{\omega,\varphi}) & := & (F,h^*)\otimes(F,h^*_\varphi)\otimes(\overline{F},h^*_\varphi)\\
(E,\langle\cdot,\cdot\rangle_{\varphi,\omega}) & := & (F,h^*_\varphi)\otimes(F,h^*_\varphi)\otimes(\overline{F},h^*).
\end{array}$$

For example the last two hermitian metrics are expressed in local coordinates by the expressions

$$\begin{array}{rcl}
\langle\alpha,\beta\rangle_{\omega,\varphi} & = & 2^{-3}h^*(dz_p,d\bar{z}_q)\,h^*_\varphi(dz_j,d\bar{z}_l)\,\overline{h^*_\varphi(dz_k,d\bar{z}_m)}\,\alpha_{pj\bar{k}}\overline{\beta_{ql\bar{m}}}\\
& = & 2^3\omega^{q\bar{p}}\omega^{l\bar{j}}_\varphi\omega^{k\bar{m}}_\varphi\alpha_{pj\bar{k}}\overline{\beta_{ql\bar{m}}},\\
\langle\alpha,\beta\rangle_{\varphi,\omega} & = & 2^3\omega^{q\bar{p}}_\varphi\omega^{l\bar{j}}_\varphi\omega^{k\bar{m}}\alpha_{pj\bar{k}}\overline{\beta_{ql\bar{m}}}
\end{array}$$

where $\alpha = \alpha_{pj\bar{k}}dz_p\otimes dz_j\otimes d\bar{z}_k$ and $\beta = \beta_{pj\bar{k}}dz_p\otimes dz_j\otimes d\bar{z}_k$. With such notations we can state the following lemma (see also [Yau]) that we will prove at the end of the section.

Lemma 6 *Let (X,ω) be a polarized Fano manifold of complex dimension n with $\omega\in 2\pi c_1$ and let $(\omega_t)_t$ be the Kähler-Ricci flow. Suppose that there exist constants k, $K>0$ such that $k^{-1}\omega\le\omega_t\le K\omega$ for all times $t\ge 0$. Then there exist constants C_1, $C_2>0$ depending only on the constants k, K and ω, such that the uniform estimate*

$$\Box_t|\nabla^{1,0}_\omega\partial\bar{\partial}\varphi_t|^2_t \ge -C_1|\nabla^{1,0}_\omega\partial\bar{\partial}\varphi_t|^2_t - C_2$$

holds for every $t\ge 0$.

By using Perelman's uniform estimate $|\dot{\varphi}_t|\le C$ and a slight modification of Yau's computation of the C^2 and C^3 uniform estimates for the complex Monge-Ampère operator (see also [Cao]), we find the following result.

Proposition 6.2 *Let X be a Fano manifold of complex dimension $n\ge 2$ and let $\omega\in 2\pi c_1(X)$ be a Kähler metric. If the Kähler-Ricci flow*

$$\dot{\varphi}_t = \log\frac{\omega^n_t}{\omega^n} + \varphi_t + c_t - h_\omega,$$

satisfies the uniform estimate $\omega_t^n \geq K_0\, \omega^n$ for some constant $K_0 > 0$ independent of $t \in [0, +\infty)$, then there exist positive constants k_0, K, $K' > 0$ independents of $t \in [0, +\infty)$, such that the uniform estimates $0 < 2n + \Delta_\omega \varphi_t \leq K$, $|\partial\bar{\partial}\varphi_t|_\omega < (K + 2\sqrt{n})/2$, $k_0^{-1}\omega < \omega_t < (K/2)\omega$ and $|\nabla_\omega^{1,0}\partial\bar{\partial}\varphi_t|_\omega \leq K'$ holds for all $t \in [0, +\infty)$.

(The C^2-uniform estimate is obvious in the case $n = 1$.)

Proof. We define the operator $\Box_t := \Delta_t - 2\frac{\partial}{\partial t}$. Consider the smooth function $A := \log(2n + \Delta_\omega \varphi_t) - k(\varphi_t + c_t)$ over $X \times [0, +\infty)$, where the constant k will be chosed later. We have the equality

$$\Box_t A = \frac{\Box_t\, \Delta_\omega \varphi_t}{2n + \Delta_\omega \varphi_t} - \frac{2|\partial \Delta_\omega \varphi_t|_t^2}{(2n + \Delta_\omega \varphi_t)^2} - k\,\Box_t \varphi_t + 2k a_t. \tag{21}$$

Set $C := \min_{x \in X} \lambda_1^\omega(x)$. Using the proposition 6.1 and the fact that $\operatorname{Tr}_\varphi \omega > 0$ we find the inequality

$$2 \operatorname{Tr}_\omega \operatorname{Ric}(\omega_\varphi) \geq -\Delta_\varphi \Delta_\omega \varphi + 4C(2n + \Delta_\omega \varphi) \operatorname{Tr}_\varphi \omega + \frac{2|\partial \Delta_\omega \varphi|_\varphi^2}{2n + \Delta_\omega \varphi}$$

which combined with the equality (21) gives

$$\Box_t A \geq -2\, \frac{\Delta_\omega \dot{\varphi}_t + \operatorname{Tr}_\omega \operatorname{Ric}(\omega_t)}{2n + \Delta_\omega \varphi_t} + 4C \operatorname{Tr}_t \omega - k\,\Box_t \varphi_t + 2k a_t, \tag{22}$$

where $\operatorname{Tr}_t \omega$ is the trace of ω respect to the metric ω_t. Taking the trace of the Kähler-Ricci flow identity $i\partial\bar{\partial}\dot{\varphi}_t = \omega_t - \operatorname{Ric}(\omega_t)$ respect to ω, we find the equality

$$\Delta_\omega \dot{\varphi}_t = 2n + \Delta_\omega \varphi_t - \operatorname{Tr}_\omega \operatorname{Ric}(\omega_t). \tag{23}$$

Moreover concerning the term $\Box_t \varphi_t$, we remark the trivial identity $\Delta_t \varphi_t = -\operatorname{Tr}_t \omega + 2n$. Using this identity with the equality (23) in the inequality (22), we find

$$\Box_t A \geq -2 + (4C + k) \operatorname{Tr}_t \omega + 2k(\dot{\varphi}_t - n + a_t). \tag{24}$$

Consider now the trivial inequality $\sum_{l=1}^n b_1 \ldots \widehat{b}_l \ldots b_n \leq (\sum_{l=1}^n b_l)^{n-1}$ for any positive number b_l. Taking the $1/(n-1)$-th power of this inequality with the terms
$b_l := 1/(1 + 2\partial^2_{l\bar{l}} \varphi_t)$ we find, in ω-orthogonal and ω_t-diagonal coordinates in a point x, the expressions

$$\begin{aligned} \frac{\operatorname{Tr}_t \omega}{4} = \sum_l \frac{1}{1 + 2\partial^2_{l\bar{l}} \varphi_t} \quad &\geq \quad \left(\frac{\sum_l \left(1 + 2\partial^2_{l\bar{l}} \varphi_t\right)}{\prod_l \left(1 + 2\partial^2_{l\bar{l}} \varphi_t\right)} \right)^{\frac{1}{n-1}} \\ &= \quad K_n\, e^{\frac{\varphi_t + c_t - h - \dot{\varphi}_t}{n-1}} (2n + \Delta_\omega \varphi)^{\frac{1}{n-1}}, \end{aligned}$$

where $K_n := 2^{\frac{-1}{n-1}} > 0$. We choose k such that $(4C + k) = 4^{-1}$ and we consider the function $u := e^A = (2n + \Delta_\omega \varphi_t) e^{-k(\varphi_t + c_t)}$. Then the previous inequality

combined with the Perelman uniform estimate $|\dot{\varphi}_t| \leq C'$ and with the estimate $|\varphi_t + c_t| \leq C$, gives

$$\frac{\operatorname{Tr}_t \omega}{4} \geq K_n e^{\frac{(1+k)(\varphi_t + c_t) - h - \dot{\varphi}_t}{n-1}} u^{\frac{1}{n-1}} \geq C_0 \, u^{\frac{1}{n-1}}$$

for some constant $C_0 > 0$ independent of t. Then the inequality (24) reduces to the inequality

$$\square_t A \geq -C_1 + C_0 \, u^{\frac{1}{n-1}}, \tag{25}$$

with $C_0, C_1 > 0$. For all $T > 0$, a point (x_0, t_0) is a maximum point for A in $X \times [0, T]$ if and only if is also a maximum point for u in $X \times [0, T]$. If $t_0 = 0$ then $u \leq C_2$ over $X \times [0, T]$, with $C_2 > 0$ independent of T. If not $\frac{\partial A}{\partial t}(x_0, t_0) \geq 0$ and $\Delta_t A(x_0, t_0) \leq 0$. Using the inequality (25) we find $u(x_0, t_0) \leq C_3$, where the constant $C_3 > 0$ is independent of T. This implies the estimate $u \leq \max\{C_2, C_3\}$ on $X \times [0, +\infty)$. So in conclusion we have found the required a priori estimate $0 < 2n + \Delta_\omega \varphi_t \leq K$. Moreover $2|\omega_t|_\omega < \operatorname{Tr}_\omega \omega_t$, since $\omega_t > 0$. This implies the required a priori estimate $|\partial\bar{\partial}\varphi_t|_\omega < (K + 2\sqrt{n})/2$. The inequality $0 < 2 + 4\partial^2_{l\bar{l}}\varphi_t < 2n + \Delta_\omega \varphi_t \leq K$ implies $\omega_t < (K/2)\omega$. By using the hypothesis we find

$$K_0 \leq \omega_t^n / \omega^n = \prod_l \left(1 + 2\partial^2_{l\bar{l}}\varphi_t\right) < (K/2)^{n-1} \left(1 + 2\partial^2_{s\bar{s}}\varphi_t\right),$$

for all s, which implies $k_0^{-1}\omega < \omega_t$ for some uniform constant $k_0 > 0$. Then by lemma 6 we deduce the estimate

$$\square_t |\nabla^{1,0}_\omega \partial\bar{\partial}\varphi_t|^2_t \geq -C_1 |\nabla^{1,0}_\omega \partial\bar{\partial}\varphi_t|^2_t - C_2 \,. \tag{26}$$

The equality (19) proved in the proposition 6.1 gives the intrinsic identity

$$2 \operatorname{Tr}_\omega \operatorname{Ric}(\omega_\varphi) = -\Delta_\varphi \Delta_\omega \varphi + 2 \operatorname{Tr}_\varphi(\omega_\varphi \cdot \operatorname{Rm}_\omega) + 4|\nabla^{1,0}_\omega \partial\bar{\partial}\varphi|^2_{\omega,\varphi},$$

where $2 \operatorname{Tr}_\varphi(\omega_\varphi \cdot \operatorname{Rm}_\omega) \geq 4\lambda_1^\omega (2n + \Delta_\omega \varphi) \operatorname{Tr}_\varphi \omega \geq -C_3$, with $C_3 > 0$. So using the identity (23), we deduce the inequality

$$\square_t \Delta_\omega \varphi_t \geq (4/k_0) |\nabla^{1,0}_\omega \partial\bar{\partial}\varphi_t|^2_t - C_4,$$

$C_4 > 0$. By taking $C_5 := k_0(C_1 + 1)/4 > 0$ we deduce by (26) the estimate

$$\square_t \left(|\nabla^{1,0}_\omega \partial\bar{\partial}\varphi_t|^2_t + C_5 \Delta_\omega \varphi_t\right) \geq |\nabla^{1,0}_\omega \partial\bar{\partial}\varphi_t|^2_t - C_6, \tag{27}$$

$C_6 > 0$. For all $T > 0$ consider a maximum point (x_0, t_0) for the function $B := |\nabla^{1,0}_\omega \partial\bar{\partial}\varphi_t|^2_t + C_5 \Delta_\omega \varphi_t$ over $X \times [0, T]$. As before we can assume $t_0 > 0$, which implies $\square_t B(x_0, t_0) \leq 0$. Then the estimate (27) implies the inequality $B(x_0, t_0) \leq C_6 + C_5 \Delta_\omega \varphi_{t_0}(x_0) \leq C_7$, for some constant $C_7 > 0$ independent of T. We deduce in conclusion the required third order uniform estimate $|\nabla^{1,0}_\omega \partial\bar{\partial}\varphi_t|^2_\omega \leq K'$, since the metrics ω_t are uniformly equivalent to the initial metric ω. $\square$

Proof of lemma 6. In order to avoid confusion with notations in the computations that will follow we will note $\varphi := \varphi_t$ and $\omega_\varphi := \omega_t$. This will apply until

equality (29). According to lemma 3 let consider $(z_1, ..., z_n)$ be ω-geodesic holomorphic coordinates of third order, with center a point x such that the metric ω_φ can be written in diagonal form in x. Explicitly $\omega = \frac{i}{2}\omega_{l\bar{r}}\, dz_l \wedge d\bar{z}_r$, where

$$\omega_{l\bar{r}} = \delta_{lr} - C_{rl}^{j\bar{k}} z_j \bar{z}_k - C_{l\bar{r}}^{pj\bar{k}} z_p z_j \bar{z}_k - \overline{C_{r\bar{l}}^{pj\bar{k}}} z_k \bar{z}_p \bar{z}_j + O(|z|^4),$$
$$2R_{j\bar{k}l\bar{r}}(x) = C_{rl}^{j\bar{k}}, \quad 2\nabla_p^{1,0} R_{j\bar{k}l\bar{r}}(x) = C_{l\bar{r}}^{pj\bar{k}}, \quad 2\nabla_{\bar{p}}^{0,1} R_{j\bar{k}l\bar{r}}(x) = \overline{C_{r\bar{l}}^{pj\bar{k}}},$$
$$\overline{C_{rl}^{j\bar{k}}} = C_{lr}^{k\bar{j}}, \quad C_{rl}^{j\bar{k}} = C_{rj}^{l\bar{k}} = C_{kl}^{j\bar{r}},$$

and the coefficients $C_{l\bar{r}}^{p,j,\bar{k}}$ are symmetric respect to the indexes p, j, l and k, r. We define $a_l := \omega_\varphi^{l\bar{l}}$. By deriving the Ricci tensor $\mathrm{Ric}(\omega_\varphi) = \mathrm{Ric}(\omega_\varphi)_{j\bar{k}}\, dz_j \wedge d\bar{z}_k$,

$$\mathrm{Ric}(\omega_\varphi)_{j\bar{k}} = -i(\partial^2_{j\bar{k}}\omega_{p\bar{r}} + 2\varphi_{jp\bar{k}\bar{r}})\omega_\varphi^{r\bar{p}} + i(\partial_j\omega_{p\bar{s}} + 2\varphi_{jp\bar{s}})\omega_\varphi^{s\bar{t}}(\partial_{\bar{k}}\omega_{t\bar{r}} + 2\varphi_{t\bar{r}\bar{k}})\omega_\varphi^{r\bar{p}},$$

we find at the point x the expression

$$\begin{aligned}
\nabla_{\omega,t}^{1,0}\,\mathrm{Ric}(\omega_\varphi)_{j\bar{k}} \;&=\; ia_p(\underbrace{C_{p\bar{p}}^{tj\bar{k}}}_{C1} - \underbrace{2\varphi_{jtp\bar{p}\bar{k}}}_{A1}) + \underbrace{4ia_p a_l \varphi_{tjp\bar{l}}\varphi_{l\bar{p}\bar{k}}}_{A2} \\
&+\; 2ia_p a_l\Big[(\underbrace{C_{lp}^{j\bar{k}}}_{C2} + \underbrace{2\varphi_{jp\bar{k}\bar{l}}}_{A3})\varphi_{tl\bar{p}} + (\underbrace{2\varphi_{tl\bar{p}\bar{k}}}_{A4} - \underbrace{C_{pl}^{t\bar{k}}}_{C3})\varphi_{jp\bar{l}}\Big] \\
&-\; 8ia_p a_l a_r(\underbrace{\varphi_{jp\bar{r}}\varphi_{tr\bar{l}}\varphi_{l\bar{p}\bar{k}}}_{A5} + \underbrace{\varphi_{jp\bar{l}}\varphi_{tr\bar{p}}\varphi_{l\bar{r}\bar{k}}}_{A6}). \qquad (28)
\end{aligned}$$

The utility of the underbraces will be discussed later. Consider now the tensor $\nabla_\omega^{1,0}\partial\bar{\partial}\varphi = \alpha_{pk\bar{l}}\, dz_p \otimes (dz_k \wedge d\bar{z}_l)$, where

$$\alpha_{pk\bar{l}} := \varphi_{pk\bar{l}} - \partial_p\, \omega_{k\bar{r}}\, \omega^{r\bar{s}}\varphi_{s\bar{l}}$$

and the derivative of its norm

$$\begin{aligned}
\partial_p|\nabla_\omega^{1,0}\partial\bar{\partial}\varphi|^2_\varphi \;&=\; 2^3\partial_p\,\omega_\varphi^{s\bar{t}}\omega_\varphi^{l\bar{j}}\,\omega_\varphi^{k\bar{r}}\,\alpha_{tj\bar{k}}\,\overline{\alpha_{sl\bar{r}}} + 2^3\,\omega_\varphi^{s\bar{t}}\partial_p\,\omega_\varphi^{l\bar{j}}\,\omega_\varphi^{k\bar{r}}\,\alpha_{tj\bar{k}}\,\overline{\alpha_{sl\bar{r}}} \\
&+\; 2^3\,\omega_\varphi^{s\bar{t}}\,\omega_\varphi^{l\bar{j}}\,\partial_p\,\omega_\varphi^{k\bar{r}}\,\alpha_{tj\bar{k}}\,\overline{\alpha_{sl\bar{r}}} + 2^3\,\omega_\varphi^{s\bar{t}}\,\omega_\varphi^{l\bar{j}}\,\omega_\varphi^{k\bar{r}}\,\partial_p\alpha_{tj\bar{k}}\,\overline{\alpha_{sl\bar{r}}} \\
&+\; 2^3\,\omega_\varphi^{st}\,\omega_\varphi^{l\bar{j}}\,\omega_\varphi^{k\bar{r}}\,\alpha_{tj\bar{k}}\,\partial_p\,\overline{\alpha_{sl\bar{r}}}
\end{aligned}$$

By using the expressions of the derivatives

$$\begin{aligned}
\partial_p\alpha_{tj\bar{k}} \;&=\; \varphi_{ptj\bar{k}} - \partial^2_{tp}\,\omega_{j\bar{a}}\,\omega^{a\bar{b}}\,\varphi_{b\bar{k}} - \partial_t\,\omega_{j\bar{a}}\,\partial_p\,\omega^{a\bar{b}}\,\varphi_{b\bar{k}} - \partial_t\,\omega_{j\bar{a}}\,\omega^{a\bar{b}}\,\varphi_{bp\bar{k}} \\
\partial_p\overline{\alpha_{sl\bar{r}}} \;&=\; \varphi_{pr\bar{s}\bar{l}} - \partial^2_{p\bar{s}}\,\omega_{a\bar{l}}\,\omega^{b\bar{a}}\,\varphi_{r\bar{b}} - \partial_{\bar{s}}\,\omega_{a\bar{l}}\,\partial_p\,\omega^{b\bar{a}}\,\varphi_{r\bar{b}} - \partial_{\bar{s}}\,\omega_{a\bar{l}}\,\omega^{b,\bar{a}}\,\varphi_{pr\bar{b}},
\end{aligned}$$

we find the the following expression for the Laplacian at the point x.

$$\begin{aligned}
\Delta_\varphi|\nabla_\omega^{1,0}\partial\bar{\partial}\varphi|^2_\varphi \;&=\; 2^5\,\omega_\varphi^{q\bar{p}}\,\Big[\partial^2_{p\bar{q}}\,\omega_\varphi^{s\bar{t}}\,\omega_\varphi^{l\bar{j}}\,\omega_\varphi^{kr}\,\varphi_{tj\bar{k}}\varphi_{r\bar{s}\bar{l}} \\
&+\; \partial_p\,\omega_\varphi^{s\bar{t}}\,\partial_{\bar{q}}\,\omega_\varphi^{l\bar{j}}\,\omega_\varphi^{k\bar{r}}\,\varphi_{tj\bar{k}}\,\varphi_{r\bar{s}\bar{l}} + \partial_p\,\omega_\varphi^{s\bar{t}}\,\omega_\varphi^{l\bar{j}}\,\partial_{\bar{q}}\,\omega_\varphi^{k\bar{r}}\,\varphi_{tj\bar{k}}\,\varphi_{r\bar{s}\bar{l}} \\
&+\; \partial_p\,\omega_\varphi^{s,\bar{t}}\,\omega_\varphi^{l\bar{j}}\,\omega_\varphi^{k\bar{r}}(\varphi_{tj\bar{q}\bar{k}} + C_{aj}^{t\bar{q}}\,\varphi_{a\bar{k}})\,\varphi_{r\bar{s}\bar{l}} \\
&+\; \partial_p\,\omega_\varphi^{s\bar{t}}\,\omega_\varphi^{l\bar{j}}\,\omega_\varphi^{k\bar{r}}\,\varphi_{tj\bar{k}}\,\varphi_{r\bar{s}\bar{l}\bar{p}} + \partial_{\bar{q}}\,\omega_\varphi^{s\bar{t}}\,\partial_p\,\omega_\varphi^{l\bar{j}}\,\omega_\varphi^{l\bar{r}}\,\varphi_{tj\bar{k}}\,\varphi_{r\bar{s}\bar{l}} \\
&+\; \omega_\varphi^{s\bar{t}}\,\partial^2_{p\bar{q}}\,\omega_\varphi^{l\bar{j}}\,\omega_\varphi^{k\bar{r}}\,\varphi_{tj\bar{k}}\,\varphi_{r\bar{t}\bar{l}} + \omega_\varphi^{s\bar{t}}\,\partial_p\,\omega_\varphi^{l\bar{j}}\,\partial_{\bar{q}}\,\omega_\varphi^{k\bar{r}}\,\varphi_{tj\bar{k}}\,\varphi_{r\bar{s}\bar{l}}
\end{aligned}$$

$$
\begin{aligned}
+ \quad & \omega_\varphi^{s\bar t}\,\partial_p\,\omega_\varphi^{l\bar j}\,\omega_\varphi^{k\bar r}\,(\varphi_{tj\bar q\bar k} + C_{sj}^{t\bar q}\,\varphi_{s\bar k})\,\varphi_{r\bar s l}\\
+ \quad & \omega_\varphi^{s\bar t}\,\partial_p\,\omega_\varphi^{l\bar j}\,\omega_\varphi^{k\bar r}\,\varphi_{tj\bar k}\,\varphi_{r\bar s\bar l\bar q} + \partial_{\bar q}\,\omega_\varphi^{s\bar t}\,\omega_\varphi^{l\bar j}\,\partial_p\,\omega_\varphi^{k\bar r}\,\varphi_{tj\bar k}\,\varphi_{r\bar s\bar l}\\
+ \quad & \omega_\varphi^{s\bar t}\,\partial_{\bar q}\,\omega_\varphi^{l\bar j}\,\partial_p\,\omega_\varphi^{k\bar r}\,\varphi_{tj\bar k}\,\varphi_{r\bar s\bar l} + \omega_\varphi^{s\bar t}\,\omega_\varphi^{l\bar j}\,\partial^2_{p\bar q}\,\omega_\varphi^{k\bar r}\,\varphi_{tj\bar k}\,\varphi_{r\bar s\bar l}\\
+ \quad & \omega_\varphi^{s\bar t}\omega_\varphi^{l\bar j}\partial_p\,\omega_\varphi^{k\bar r}(\varphi_{tj\bar k\bar q} + C_{sj}^{t\bar q}\varphi_{s\bar k})\varphi_{r\bar s\bar l} + \omega_\varphi^{s\bar t}\omega_\varphi^{l\bar j}\partial_p\,\omega_\varphi^{k\bar r}\varphi_{tj\bar k}\varphi_{r\bar s\bar l\bar q}\\
+ \quad & \partial_{\bar q}\,\omega_\varphi^{s,\bar t}\,\omega_\varphi^{l\bar j}\,\omega_\varphi^{k\bar r}\,\varphi_{ptj\bar k}\,\varphi_{r\bar s\bar l} + \omega_\varphi^{s,\bar t}\,\partial_{\bar q}\,\omega_\varphi^{l\bar j}\,\omega_\varphi^{k\bar r}\,\varphi_{ptj\bar k}\,\varphi_{r\bar s\bar l}\\
+ \quad & \omega_\varphi^{s\bar t}\,\omega_\varphi^{l\bar j}\,\partial_{\bar q}\,\omega_\varphi^{k\bar r}\,\varphi_{ptj\bar k}\,\varphi_{r\bar s\bar l}\\
+ \quad & \omega_\varphi^{s\bar t}\,\omega_\varphi^{lj}\,\omega_\varphi^{k\bar r}(\varphi_{ptj\bar k\bar q} + C_{j\bar a}^{tp\bar q}\,\varphi_{a\bar k} + C_{aj}^{t\bar q}\,\varphi_{ap\bar k})\,\varphi_{r\bar s\bar l}\\
+ \quad & \omega_\varphi^{s\bar t}\,\omega_\varphi^{l\bar j}\,\omega_\varphi^{k\bar r}\varphi_{ptj\bar k}\,\varphi_{r\bar s\bar l\bar q} + \partial_{\bar q}\,\omega_\varphi^{s\bar t}\,\omega_\varphi^{l\bar j}\,\omega_\varphi^{k\bar r}\varphi_{tj\bar k}(\varphi_{pr\bar s\bar l} + C_{la}^{p\bar s}\varphi_{r\bar a})\\
+ \quad & \omega_\varphi^{s\bar t}\,\partial_{\bar q}\,\omega_\varphi^{l\bar j}\,\omega_\varphi^{k\bar r}\,\varphi_{tj\bar k}\,(\varphi_{pr\bar s\bar l} + C_{la}^{p\bar s}\,\varphi_{r\bar a})\\
+ \quad & \omega_\varphi^{s\bar t}\,\omega_\varphi^{l\bar j}\,\partial_{\bar q}\,\omega_\varphi^{k\bar r}\,\varphi_{tj\bar k}(\varphi_{pr\bar s\bar l} + C_{la}^{p\bar s}\,\varphi_{r\bar a})\\
+ \quad & \omega_\varphi^{s\bar t}\,\omega_\varphi^{l\bar j}\,\omega_\varphi^{k\bar r}\,(\varphi_{tj\bar k\bar q} + C_{aj}^{t\bar q}\,\varphi_{a\bar k})\,(\varphi_{pr\bar s\bar l} + C_{la}^{p\bar s}\,\varphi_{r\bar a})\\
+ \quad & \omega_\varphi^{s\bar t}\,\omega_\varphi^{lj}\,\omega_\varphi^{k\bar r}\,\varphi_{tj\bar k}\,\Big(\varphi_{pr\bar s\bar l\bar q} + \overline{C_{l\bar a}^{tq\bar p}}\,\varphi_{r\bar a} + C_{la}^{p\bar t}\,\varphi_{r\bar a\bar q}\Big)\Big].
\end{aligned}
$$

Then using the expressions $\partial_{\bar l}\,\omega_\varphi^{s\bar t} = -2a_s a_t\,\varphi_{s l\bar t}$ and

$$
\partial^2_{k\bar l}\,\omega_\varphi^{s\bar t} = a_s a_t\left[C_{ts}^{k\bar l} - 2\varphi_{ks\bar l\bar t} + 4a_r\,(\varphi_{ks\bar r}\varphi_{r\bar l\bar t} + \varphi_{kr\bar t}\,\varphi_{s\bar l\bar r})\right],
$$

at the point x of the derivatives of the inverse matrixs we find the following expression for the Laplacian at the point x.

$$
\begin{aligned}
\Delta_\varphi|\nabla^{1,0}_\omega\partial\bar\partial\varphi|^2_\varphi \quad = \quad & 2^5 a_p a_t a_j a_k\Big\{\\
& a_l[4a_r(\varphi_{pl\bar r}\varphi_{r\bar p\bar t} + \underbrace{\varphi_{pr\bar t}\varphi_{l\bar p\bar r}}_{B1d}) - 2\varphi_{pl\bar p\bar t} + C_{tl}^{p\bar p}]\varphi_{tj\bar k}\varphi_{k\bar j\bar l}\\
+ \quad & 4a_l a_r(\underbrace{\varphi_{k\bar r\bar l}\varphi_{l\bar p\bar j}}_{A5} + \underbrace{\varphi_{l\bar r\bar j}\varphi_{k\bar p\bar l}}_{B1d})\varphi_{pr\bar t}\varphi_{tj\bar k}\\
- \quad & 2a_l(\underbrace{\varphi_{tj\bar p\bar k}}_{\overline{A4}} + C_{kj}^{t\bar p}\varphi_{k\bar k})\varphi_{pl\bar t}\varphi_{k\bar l\bar j} - \underbrace{2a_l\varphi_{k\bar l\bar j\bar p}\varphi_{pl\bar t}\varphi_{tj\bar k}}_{B1b}\\
+ \quad & \underbrace{4a_l a_r\varphi_{pl\bar j}\varphi_{tj\bar k}\varphi_{k\bar r\bar l}\varphi_{r\bar p\bar t}}_{B1c}\\
+ \quad & a_l[4a_r(\varphi_{pl\bar r}\varphi_{r\bar p\bar j} + \underbrace{\varphi_{pr\bar j}\varphi_{l\bar p\bar r}}_{B1c}) - 2\varphi_{pl\bar p\bar j} + C_{jl}^{p\bar p}]\varphi_{tj\bar k}\varphi_{k\bar t\bar l}\\
+ \quad & 4a_l a_r(\underbrace{\varphi_{pl\bar j}\varphi_{k\bar p\bar r}}_{A6} + \underbrace{\varphi_{pk\bar r}\varphi_{l\bar p\bar j}}_{\overline{A5}})\varphi_{tj\bar k}\varphi_{r\bar t\bar l}\\
- \quad & 2a_l(\underbrace{\varphi_{tj\bar p\bar k}}_{A4} + C_{kj}^{t\bar p}\varphi_{k\bar k})\varphi_{pl\bar j}\varphi_{k\bar t\bar l} - \underbrace{2a_l\varphi_{k\bar t\bar l\bar p}\varphi_{pl\bar j}\varphi_{tj\bar k}}_{B1a}\\
+ \quad & \underbrace{4a_l a_r\varphi_{pk\bar r}\varphi_{tj\bar k}\varphi_{r\bar l\bar t}\varphi_{l\bar p\bar j}}_{\overline{A6}}\\
+ \quad & a_l[4a_r(\varphi_{pk\bar r}\varphi_{r\bar p\bar l} + \underbrace{\varphi_{pr\bar l}\varphi_{k\bar p\bar r}}_{B2a}) - 2\varphi_{pk\bar p\bar l} + C_{lk}^{p\bar p}]\varphi_{tj\bar k}\varphi_{l\bar t\bar j}
\end{aligned}
$$

$$
\begin{aligned}
- &\quad \underbrace{2a_l(\varphi_{tj\bar p\bar k} + C^{t\bar p}_{kj}\varphi_{k\bar k})\varphi_{pk\bar l}\varphi_{l\bar t\bar j}}_{B2} - \underbrace{2a_l\varphi_{l\bar t\bar j\bar p}\varphi_{pk\bar l}\varphi_{tj\bar k}}_{\overline{A2}} \\
- &\quad \underbrace{2a_l\varphi_{ptj\bar k}\varphi_{l\bar p\bar t}\varphi_{k\bar j\bar l}}_{B1} - \underbrace{2a_l\varphi_{ptj\bar k}\varphi_{l\bar p\bar j}\varphi_{k\bar t\bar l}}_{A2} - \underbrace{2a_l\varphi_{ptj\bar k}\varphi_{k\bar p\bar l}\varphi_{l\bar t\bar j}}_{B1} \\
+ &\quad (\underbrace{\varphi_{ptj\bar k\bar p}}_{A1} + C^{t\bar p}_{rj}\varphi_{rp\bar k} + \underbrace{C^{tp\bar p}_{j\bar k}\varphi_{k\bar k}}_{C1})\varphi_{k\bar t\bar j} + \underbrace{\varphi_{ptj\bar k}\varphi_{k\bar p\bar t\bar j}}_{B1} \\
- &\quad 2a_l(\underbrace{\varphi_{pk\bar l\bar j}}_{\overline{A3}} + C^{p,\bar l}_{jk}\varphi_{k\bar k})\varphi_{tj\bar k}\varphi_{l\bar p\bar t} \\
- &\quad 2a_l(\underbrace{\varphi_{pk\bar t\bar l}}_{A3} + C^{p\bar t}_{lk}\varphi_{k\bar k})\varphi_{tj\bar k}\varphi_{l\bar p\bar j} \\
- &\quad \underbrace{2a_l(\varphi_{pl\bar t\bar j} + C^{p\bar t}_{jl}\varphi_{l\bar l})\varphi_{tj\bar k}\varphi_{k\bar p\bar l}}_{B2a} \\
+ &\quad \underbrace{(\varphi_{tj\bar p\bar k} + C^{t\bar p}_{kj}\varphi_{k\bar k})(\varphi_{pk\bar t\bar j} + C^{p\bar t}_{jk}\varphi_{k\bar k})}_{B2} \\
+ &\quad (\underbrace{\varphi_{pk\bar t\bar j\bar p}}_{\overline{A1}} + C^{p\bar t}_{jr}\varphi_{k\bar r\bar p} + \underbrace{\overline{C^{tp\bar p}_{j\bar k}}\varphi_{k\bar k}}_{\overline{C1}})\varphi_{tj\bar k}\Big\}
\end{aligned}
$$

We explain now the meaning of the underbraces. Set $a_{ptjk} := a_p a_t a_j a_k$ and

$$
B1 := 2^5 a_{ptjk} Big\{\Big[\varphi_{ptj\bar k} - 2\sum_l a_l\left(\varphi_{pt\bar l}\varphi_{lj\bar k} + \varphi_{pl\bar k}\varphi_{tj\bar l}\right)\Big] \times [\text{conjugate}]\Big\} \geq 0
$$

$$
B2 := 2^5 a_{ptjk}\Big\{\Big(\varphi_{tj\bar p\bar k} + C^{t\bar p}_{kj}\varphi_{k\bar k} - 2\sum_l a_l\varphi_{tj\bar l}\varphi_{l\bar p\bar k}\Big) \times (\text{conjugate})\Big\} \geq 0\,.
$$

Then the underbraced terms in the previous expression of the Laplacian corresponds to the terms $A*, C*$ of the expression (28) of the covariant derivative of the Ricci tensor and the terms $B*$ just defined. To be more precise to see those correspondences we need to make the following change of indexes of the underbraced terms of the Laplacian.

A2	$(k,l,p,t) \to (l,k,t,p)$		
A3	$(k,l,p,t,j) \to (p,k,j,l,t)$	$\overline{\text{A3}}$	$(l,p) \to (p,l)$
A4	$(l,j) \to (j,l)$	$\overline{\text{A4}}$	$(t,j,k,l,p) \to (k,p,l,j,t,)$
A5	$(k,l,p,j,t,r) \to (l,k,j,t,r,p)$	$\overline{\text{A5}}$	$(l,r) \to)$
A6	$(k,j,r,t,l) \to (l,r,k,j,t)$	$\overline{\text{A6}}$	$(r,l,t,j) \to (l,r,j,t)$
B1a	$(t,j,p,l) \to (p,l,t,j)$	B1b	$(t,l) \to (l,t)$
B1c	$(j,l,t,p) \to (l,j,p,t)$	B1d	$(t,r,l) \to (l,t,r)$
B2a	$(l,k) \to (k,l)$		

Using the expression of the Ricci tensor at the point x

$$
\mathrm{Ric}(\omega_\varphi)_{j\bar k} = -ia_p(C^{j\bar k}_{pp} + 2\varphi_{jp\bar k\bar p}) + 4ia_p a_t\varphi_{jp\bar t}\varphi_{t\bar p\bar k}\,,
$$

and the expression (28) of the covariant derivative of the Ricci tensor at the point x we find the expression

$$
\Delta_\varphi|\nabla^{1,0}_\omega\partial\bar\partial\varphi|^2_\varphi = -2^5 i a_{ltjk}\Big[\mathrm{Ric}(\omega_\varphi)_{l\bar t}\,\varphi_{k\bar j\bar l} + \mathrm{Ric}(\omega_\varphi)_{l\bar j}\,\varphi_{k\bar t\bar l} + \mathrm{Ric}(\omega_\varphi)_{k\bar l}\,\varphi_{l\bar t\bar j}\Big]\varphi_{tj\bar k}
$$

$$
\begin{aligned}
&+ \quad 4\Re e\Big[i\left\langle \nabla^{1,0}_{\omega}\operatorname{Ric}(\omega_{\varphi}), \nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi\right\rangle_{\varphi} + 2i\left\langle \operatorname{Tr}_{\varphi}\nabla^{1,0}_{\omega}R_{\omega}, \nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi\right\rangle_{\varphi,\omega}\Big]\\
&+ \quad 2^5 a_{ptjk}\, 2\Re e\Big[a_l\Big(\underbrace{C^{j\bar{k}}_{lp}\,\varphi_{tl\bar{p}}}_{C2} - \underbrace{C^{t\bar{k}}_{pl}\,\varphi_{jp\bar{l}}}_{C3}\Big)\varphi_{k\bar{t}\bar{j}} - 4a_l C^{t\bar{p}}_{kj}\,\varphi_{k\bar{k}}\,\varphi_{pl\bar{t}}\,\varphi_{k\bar{l}\bar{j}}\\
&+ \quad C^{t\bar{p}}_{rj}\,\varphi_{rp\bar{k}}\,\varphi_{k\bar{t}\bar{j}}\Big] + B1 + B2\,, \qquad (29)
\end{aligned}
$$

where $\Re e$ is the real part of a complex number and

$$(\operatorname{Tr}_{\varphi}\nabla^{1,0}_{\omega}R_{\omega})(\xi,\eta,\mu) := \operatorname{Tr}_{\varphi}[\nabla^{1,0}_{\omega,\xi}R_{\omega}(\eta,\mu,\cdot,\cdot)] = \operatorname{Tr}_{\varphi}[\nabla^{1,0}_{\omega,\xi}R_{\omega}(\cdot,\cdot,\eta,\mu)]\,,$$

for all $\xi,\eta \in T^{1,0}_{X,J}$, $\mu \in T^{0,1}_{X,J}$. From now on we reconsider our original notations $\varphi_t = \varphi$ and $\omega_t = \omega_{\varphi}$. Using the fact that the inverse matrix $(\omega_t^{k,\bar{l}})_{k,l}$ evolves by the formula

$$\frac{d}{dt}\omega_t^{k,\bar{l}} = -\omega_t^{k,\bar{l}} + 2\omega_t^{k,\bar{j}} R_{j,\bar{p}}(t)\,\omega_t^{p,\bar{l}},$$

where $\operatorname{Ric}(\omega_t) = iR_{j,\bar{p}}(t)\,dz_j\wedge d\bar{z}_p$, we find at the point x the expression

$$
\begin{aligned}
\frac{\partial}{\partial t}|\nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi_t|^2_t &= -3|\nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi_t|^2_t + 2^4 a_{pljk}\Big[R_{l\bar{p}}(t)\,\varphi_{k\bar{j}\bar{l}} + R_{l\bar{j}}(t)\,\varphi_{k\bar{l}\bar{p}}\\
&+ \quad R_{k\bar{l}}(t)\,\varphi_{l\bar{p}\bar{j}}\Big]\varphi_{pj\bar{k}} + 2\Re e\left\langle \nabla^{1,0}_{\omega}\partial\bar{\partial}\dot{\varphi}_t, \nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi_t\right\rangle_t. \qquad (30)
\end{aligned}
$$

Using the expression (29) and the fact that all the metrics ω_t are uniformly equivalents to the initial metric ω, we obtain the inequality

$$
\begin{aligned}
&\Delta_t|\nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi_t|^2_t \geq 2^5 a_{ptjk}\Big[R_{l\bar{p}}(t)\,\varphi_{k\bar{j}\bar{l}} + R_{l\bar{j}}(t)\,\varphi_{k\bar{l}\bar{p}} + R_{k\bar{l}}(t)\,\varphi_{l\bar{p}\bar{j}}\Big]\varphi_{pj\bar{k}}\\
&+4\Re e\left\langle i\nabla^{1,0}_{\omega}\operatorname{Ric}(\omega_t), \nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi_t\right\rangle_t - C_1|\nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi_t|^2_t - C_2'|\nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi_t|_t, \qquad (31)
\end{aligned}
$$

where C_1, $C_2' > 0$ are two constants independents of t. By deriving the Kähler-Ricci flow identity $i\partial\bar{\partial}\dot{\varphi}_t = \omega_t - \operatorname{Ric}(\omega_t)$, we find the equality

$$i\nabla^{1,0}_{\omega}\partial\bar{\partial}\dot{\varphi}_t = i\nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi_t - \nabla^{1,0}_{\omega}\operatorname{Ric}(\omega_t),$$

which combined with the relations (30) and (31) gives the uniform estimate

$$\Box_t|\nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi_t|^2_t \geq (2-C_1)|\nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi_t|^2_t - C_2'|\nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi_t|_t \geq -C_1|\nabla^{1,0}_{\omega}\partial\bar{\partial}\varphi_t|^2_t - C_2\,,$$

for some uniform constant $C_2 > 0$ sufficiently big. □

7 The existence of a Kähler-Einstein metric

We remind that the uniform estimate $\omega_t^n \geq k\,\omega^n$ is equivalent to the uniform estimate $\varphi_t + c_t \leq C$. Then the identity 14 implies that the K-energy is also uniformly bounded from below along the flow, thus the limit $\lim_{t\to+\infty}\nu_{\omega}(\varphi_t)$ is finite. We remind also that along the Kähler-Ricci flow we have the identity

$$\frac{d}{dt}\nu_{\omega}(\varphi_t) = -\int\!\!\!\!\!\!-_{X} |\partial\dot{\varphi}_t|^2_t\,\omega_t^n\,.$$

So for all increasing sequences of times $(\tau_k) \subset [0, +\infty)$, $\tau_k \to +\infty$ there exist a sequence $(t_k)\,, t_k \in [\tau_k, \tau_{k+1}]$ such that

$$\lim_{k \to +\infty} \int\!\!\!\!\!\!-_X |\partial \dot{\varphi}_{t_k}|^2_{t_k}\, \omega^n_{t_k} = 0. \tag{32}$$

Moreover the C^2 and C^3-uniform estimates $|\partial\bar{\partial}\varphi_t|_{C^0(X)}$, $|\nabla^{1,0}_\omega \partial\bar{\partial}\varphi_t|_{C^0(X)} \leq C$ implient that the $(1,1)$-formes $(\partial\bar{\partial}\varphi_t)_{t\in[0,+\infty)}$ are uniformly bounded in the $C^\alpha(X)$-topology. The operator Δ_t is uniformly elliptic with coefficients uniformly bounded in C^α-norm, at least. The right hand side of the equation (11)

$$\square_t(\xi.\varphi_t) + 2\xi.\varphi_t = (\mathrm{Tr}_\omega - \mathrm{Tr}_t)(L_\xi\, \omega) + 2\xi.h_\omega\,,$$

$\xi \in \mathcal{E}(T_X)(U)$, is also uniformly bounded in C^α-norm, at least. By the regularity theory for parabolic equations [Lad] we deduce that the functions $(\xi.\varphi_t)_{t\in[0,+\infty)}$ are uniformly bounded in $C^{2,\alpha}$-norm. Then the C^0-uniform estimate $|\varphi_t + c_t| \leq C$ implies the existence of a subsequence (s_k) of (t_k) such that the sequences $(\varphi_{s_k} + c_{s_k})$, $(d\varphi_{s_k})$, $(\partial\bar{\partial}\varphi_{s_k})$ and $(\nabla^{1,0}_\omega \partial\bar{\partial}\varphi_{s_k})$ convergent uniformly respectively to φ_∞, $d\varphi_\infty$, $\partial\bar{\partial}\varphi_\infty$, and $\nabla^{1,0}_\omega\partial\bar{\partial}\varphi_\infty$. The uniform estimate $\omega^n_t/\omega^n \geq K_0 > 0$ gives $\omega^n_{\varphi_\infty}/\omega^n \geq K_0 > 0$, which implies $i\partial\bar{\partial}\varphi_\infty > -\omega$. Moreover we deduce the existence of the limits

$$\psi := \lim_{k\to+\infty} \dot{\varphi}_{s_k} = \log \frac{\omega^n_{\varphi_\infty}}{\omega^n} + \varphi_\infty - h_\omega$$

and $\partial\psi = \lim_{k\to+\infty} \partial\dot{\varphi}_{s_k}$ in the topology of the uniform convergence at least. Then the limit (32) implies

$$0 = \lim_{k\to+\infty} \int\!\!\!\!\!\!-_X |\partial\dot{\varphi}_{s_k}|^2_{s_k}\, \omega^n_{s_k} = \int\!\!\!\!\!\!-_X |\partial\psi|^2_{\varphi_\infty} \omega^n_{\varphi_\infty},$$

which means $\psi = 0$, by the integral normalization of $\dot{\varphi}_t$. So we have a solution $\varphi_\infty \in \mathcal{P}^{3,\alpha}_\omega$ of the elliptic non-linear equation

$$F(\varphi_\infty) := \log \frac{\omega^n_{\varphi_\infty}}{\omega^n} + \varphi_\infty - h_\omega = 0.$$

The ellipticity follows from the fact that $i\partial\bar{\partial}\varphi_\infty > -\omega$ and the expression of the differential $d_{\varphi_\infty}F(v) = 2^{-1}\Delta_{\varphi_\infty} v + v$. By Schauder elliptic regularity (see [Aub], Th. 3.56, pag. 86) we deduce that the solution φ_∞ is smooth. In conclusion we have solve the Einstein equation $\mathrm{Ric}(\omega_{\varphi_\infty}) = \omega_{\varphi_\infty}$. Clearly the Einstein metric ω_{φ_∞} is G-invariant if the initial metric ω of the Kähler-Ricci flow is G-invariant. $\square$

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